# Scarf's algorithm and stable marriages 

Yuri Faenza, Chengyue He ${ }^{\dagger}$ and Jay Sethuraman ${ }^{\ddagger}$<br>Columbia University

March 3, 2023


#### Abstract

Scarf's algorithm gives a pivoting procedure to find a special vertex-a dominating vertex-in down-monotone polytopes. This paper studies the behavior of Scarf's algorithm when employed to find stable matchings in bipartite graphs. First, it proves that Scarf's algorithm can be implemented to run in polynomial time, showing the first positive result on its runtime in significant settings. Second, it shows an infinite family of instances where, no matter the pivoting rule and runtime, Scarf's algorithm outputs a matching from an exponentially small subset of all stable matchings, thus showing a structural weakness of the approach.


## 1 Introduction

The theory of stable matchings has been studied for decades by the algorithms and operations research community. This effort has led to a variety of algorithms, which often give complementary approaches to the same problem. For instance, when we are given weights on the edges, a stable matching of maximum total weight in a bipartite (marriage) instance can be found using a combinatorial algorithm [22], any linear programming solver [33, 35, 42], or an interplay of the two [18]. The problem of finding a stable matching in a marriage instance can then also be solved via multiple algorithms, including Gale and Shapley's Deferred Acceptance algorithm [19], mechanisms using compensation chains [14], and more (see [27] for extensive references).

Scarf's lemma 37 provides yet another algorithm for finding a stable matching in a marriage instance, with a number of distinct features that call for a deeper understanding. First, it is geometric in nature, while most other algorithms are combinatorial. Secondly, it applies under more general conditions, that go well beyond the classical marriage setting by Gale and Shapley. For instance, it has been used to design heuristics or (not necessarily polynomial-time) exact algorithms for matching markets with complex side constraints for which no alternative algorithms are known; examples of such side constraints are those arising from the presence of couples or budgets or the need to meet a proportionality requirement [7, 28, 29, 30]. Scarf's lemma has also been employed to show the existence of objects arising in the theory of graphs, matroids, posets, and games [1, 2, 7]. Its connections with Sperner's lemma [25] and Nash equilibria 43] have also been studied.

In essence, Scarf's lemma guarantees the existence of a vertex of a down-monotone polytope that is dominating (see Section 1.1 for definitions). The original proof by Scarf is algorithmic, as

[^0]it is based on a pivoting rule that, starting from a certain vertex of the polytope, is guaranteed to terminate at a dominating vertex. When applied to the bipartite matching polytope, it guarantees the existence of a stable matching in a marriage instance. When applied to the fractional matching polytope, and combined with its half-integrality, it guarantees the existence of a stable partition in a roommate instance, a result originally obtained by Tan [38] via a combinatorial argument. As discussed earlier, however, Scarf's lemma applies much more generally: for instance, when applied to a fractional hypergraph matching polytope, it establishes the existence of a fractional stable matching in a hypergraph [2]. Many allocation problems in which integral solutions may not exist can be modelled as the problem of finding a fractional stable matching in a hypergraph [7], which is then rounded to an integer, almost-feasible solution [29].

The generality of Scarf's result comes however at a computational price: the problem of finding a dominating vertex of a down-monotone polytope is PPAD-Complete [24]. In particular, it remains PPAD-Complete on the hypergraph matching polytope [24], even in quite restricted settings [12, 23].

These negative results frustrate the search for a proof of polynomial-time convergence of Scarf's algorithm in its most general terms. However, the broad applicability of Scarf's lemma calls for a more fine-grained analysis of the algorithm. To the best of our knowledge, in no relevant case has the polynomial-time convergence of Scarf's algorithm been established so far. This is in stark contrast with the many positive results known for other important geometric algorithms in combinatorics and combinatorial optimization, including the cutting plane method [10] and pivoting procedures such as the simplex method [8, 31, 40].

The goal of this paper is to shed some light on the strengths and limits of Scarf's algorithm, by focusing on its application to the marriage model. The restriction to this model is motivated by multiple reasons. First, the marriage model is arguably one of the most relevant settings where Scarf's lemma holds true, and applications of Scarf's algorithm in market design rely on extensions of the marriage model [29, 30, 28]. For some of those applications, we do not know if Scarf's algorithm can be implemented to run in polynomial time, and understanding their common special case is a natural first step. Second, computational experiments have shown that Scarf's algorithm can be used as a heuristic in some of those markets [7], hence understanding which underlying structure implies fast running time is an intriguing and important question. In particular, Biro and Fleiner [6] pose many open questions on the features of Scarf's algorithm when employed to find stable matchings. Some of these open questions are answered in this paper, see Section 1.2 and Section 1.3. Last, matching problems have often proved to be at the "proper level of difficulty" [26] for developing tractable, yet non-trivial, theories and algorithms.

Before delving into the details, we formally introduce Scarf's lemma and its associated algorithm in the next section and review them more in detail in Section 3. Our results are presented formally in Section 1.2, and related literature is discussed in Section 1.3. An overview of the techniques used to show our results is given in Section 2, while full details are carried out in Sections 5, 6, and 7.

### 1.1 Scarf's Lemma and Applications

Consider a down-monotone polytope $P \subset \mathbb{R}^{(n+m)}$ in standard form, i.e.,

$$
\begin{equation*}
P=\left\{x \in \mathbb{R}_{\geq 0}^{n+m}: A x=b\right\}, \text { with } A=\left(I \mid A^{\prime}\right) \tag{1}
\end{equation*}
$$

where $I$ is the $n \times n$ identity matrix, $A \in \mathbb{Q}_{\geq 0}^{n \times(n+m)}, A^{\prime}=\left(a_{i, j}^{\prime}\right)$, and $b \in \mathbb{Q}_{+}^{n}$. We say that $A$ as above is in standard form. We call a basis (in the classical linear algebra sense) $B$ for $A$ that is feasible for (11) an $(A, b)$ basis. The definition depends on $b$ to emphasize feasibility.

A matrix $C=\left(c_{i, j}\right) \in \mathbb{Z}^{n \times(n+m)}$ is an ordinal matrix if it has distinct entries that satisfy $c_{i, i}<c_{i, k}<c_{i, j}$ for any $i \neq j \in[n], k \in[n+m] \backslash[n]$, where we let $[n]=\{1, \ldots, n\}$. As we see later,
the only relevant information contained in $C$ is the relative order of entries in each row of $C$, so we assume w.l.o.g. $c_{i j}=O(n+m)$ for all $i, j$.

Consider a set $D=\left\{j_{1}, \ldots, j_{n}\right\}$ of $n$ columns of $C$. For every row $i \in[n]$, define the minimum element

$$
\begin{equation*}
u_{i}=\min _{k \in[n]} c_{i, j_{k}} . \tag{2}
\end{equation*}
$$

Fix $j \in D$. By definition, $u_{i} \leq c_{i, j}$ for any row index $i$. We succinctly write this set of relations by $u \leq c_{j}$. The set $D$ is called an ordinal basis of $C$ if for every column $h \in[m]$, there is at least one $i \in[n]$ such that $u_{i} \geq c_{i h}$. The associated vector $u \in \mathbb{Q}^{n}$ is called the utility vector of this ordinal basis. These concepts are illustrated in the example below.

Example 1. The matrix, $C$, below is an ordinal matrix and $D=\{4,5,6\}$ is an ordinal basis with $u=(1,1,1)^{T}$.

$$
C=\left(\begin{array}{ccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 5 & 4 & 2 & 3 & 1 \\
5 & 0 & 4 & 1 & 2 & 3 \\
5 & 4 & 0 & 3 & 1 & 2
\end{array}\right) .
$$

An $(A, b)$ basis that is also an ordinal basis of $C$ is called a dominating basis for $(A, b, C)$. Given a dominating basis $B$ for $(A, b, C)$, the unique vertex $x$ of (11) corresponding to $B$ is called a dominating vertex for $(A, b, C)$. Scarf's lemma [37], presented next, shows that such a vertex always exists.

Theorem 2 (Scarf's Lemma). Let $A \in \mathbb{Q}_{\geq 0}^{n \times(n+m)}$ be in standard form, $C \in \mathbb{Z}^{n \times(n+m)}$ be an ordinal matrix, and $b \in \mathbb{Q}_{+}^{n}$ such that (11) is bounded. Then there exists a dominating vertex for $(A, b, C)$.

Theorem 2 was originally proved using a pivoting algorithm, described next.

Scarf's algorithm. Scarf's algorithm starts by letting $B=\{1, \cdots, n\}$ and $D=\left\{j_{0}, 2,3, \cdots, n\right\}$, where $j_{0}$ is selected from the columns $k \in\{n+1, \ldots, m\}$ so as to maximize $c_{1 k}$. That is, $c_{1 j_{0}}=$ $\max _{k>n} c_{1 k}$ (in Example 1, $j_{0}=5$ and $D=\{5,2,3\}$ ). Note that $|B \cap D| \geq n-1, B$ is an $(A, b)$ basis, and $D$ is an ordinal basis of $C$.

These properties are satisfied throughout the algorithm: at the beginning of each iteration, we have an $(A, b)$ basis $B$ and an ordinal basis $D$ with $|B \cap D| \geq n-1$. If $|B \cap D|=n$, then $B=D$ and the algorithm halts and outputs the dominating basis $B$. Note that, in this case, $B$ is a dominating basis for $(A, b, C)$. Else, we let $\left\{j_{t}\right\}=D \backslash B$ and perform the following:

1 Cardinal pivot: A column $j_{\ell} \in B \cap D$ is chosen, so that $B^{\prime}:=B \backslash\left\{j_{\ell}\right\} \cup\left\{j_{t}\right\}$ is a basis for $(A, b)$. That is, $j_{t}$ enters and $j_{\ell}$ leaves $B$.

2 Ordinal pivot: A column $j^{*} \notin D$ is chosen, so that $D^{\prime}:=D \backslash\left\{j_{\ell}\right\} \cup\left\{j^{*}\right\}$ is an ordinal basis of $C$. That is, $j_{\ell}$ leaves and $j^{*}$ enters $D$.

The algorithm then proceeds to the next iteration, setting $B=B^{\prime}$ and $D=D^{\prime}$. Note that the change of bases in cardinal pivoting coincides with the classical pivoting operation employed by, e.g., the simplex algorithm. In particular, by basic linear algebra, there is always at least a feasible choice for $j_{\ell}$ [5]-even though, if the polytope is degenerate, multiple choices may be possible. Ordinal pivoting is, at every step, uniquely determined [37].

In non-degenerate polytopes, Scarf [37] proved that the algorithm always terminates. For degenerate polytopes, a reduction to the non-degenerate case shows that a pivoting rule leading to
convergence exists, but to the best of our knowledge, this pivoting rule is polytope-specific and does not match any of the "standard" pivoting rules investigated, e.g., in the literature on the simplex method (9).

Applications to Stable Marriage. As mentioned in the introduction, when specialized to certain polytopes, Scarf's lemma can be used to establish the existence of specific combinatorial objects. In this section, we review the setting that is relevant for our paper, that is, the stable marriage model.

Let $M=\left\{m_{1}, \ldots, m_{k}\right\}$ denote a set of men, and $W=\left\{w_{1}, \ldots, w_{k}\right\}$ denote a set of women. Without loss of generality, we assume that $|M|=|W|=k$ and that every possible pair ( $m_{i}, w_{j}$ ) is acceptable: for any man $m \in M$, there is a strict linear order over $W \cup\{m\}$ such that $m$ is ordered last ( $m$ being matched to $m$ means that $m$ is left unmatched). It is well-known that every stable marriage instance can be transformed to an instance with these properties without loss of generality [20]. We refer to this order as the preference order of $m$ and denote it by $\succ_{m}$. The preference order for every woman $w$ is defined analogously and it is denoted by $\succ_{w}$. A matching is a set of disjoint pairs $\mu \subset M \times W$. Given a matching $\mu$, a pair $(m, w) \in M \times W$ is a blocking pair if both $w \succ_{m} \mu(m)$ and $m \succ_{w} \mu(w)$, where for $v \in M \cup W$, we denote by $\mu(v)$ the partner of $v$ in $\mu$ if such a partner exists, or $\mu(v)=v$ otherwise. A matching $\mu$ is called stable if no blocking pair exists. It has been observed, e.g., in Biró and Fleiner [7], that Scarf's lemma can be used to show the existence of stable matchings in marriage instances.

Theorem 3. Consider an instance $\mathcal{I}$ of the marriage model defined over a bipartite graph $G(V, E)$, and let (1) describe the (classical) matching polytope of $G(V, E)$, with $A^{\prime}$ being the node-edge incidence matrix of $G$ and every component of $b$ being 1. There exists a matrix $C$ such that every dominating vertex for $(A, b, C)$ is the characteristic vector of a stable matching.

For $A, b, \mathcal{I}$ as in Theorem 3, we say that $A, b$ are induced by instance $\mathcal{I}$.

### 1.2 Our Contributions

Polynomiality of Scarf's Algorithm in the Marriage Model. As our first result, we show that Scarf's algorithm on an input $\left(A, b, C^{*}\right)$, where $A, b$ are as in Theorem 3 and $C^{*}$ is a specific choice among the matrices $C$ that make Theorem 3 true, can be implemented to run in polynomial time. In particular, as the matching polytope is degenerate, we find a pivoting rule (see Algorithm (1) to control the cardinal pivots. Biró and Fleiner [6] ask whether Scarf's algorithm terminates in polynomial time for matching games, and our result gives a positive answer to their question in the special case of stable marriage games.

Theorem 4. For any instance $\mathcal{I}$ of the marriage model over a bipartite graph $G(V, E)$, there exists a cardinal pivoting rule such that Scarf's algorithm runs in polynomial time on input $\left(A, b, C^{*}\right)$, where $\left\{x \in \mathbb{R}_{\geq 0}^{m}: A x=b\right\}$ describes the matching polytope of $G$ and $C^{*}$ is a specific matrix that make Theorem ${ }^{3}$ true. In particular, the output will be the characteristic vector of a stable matching.

As a building block to the proof of Theorem 4, we develop an understanding of pivoting operations connecting ordinal and feasible bases that may be visited by Scarf's algorithm.

Polynomial-time Convergence Through a Perturbation of the Polytope. We show that polynomial-time convergence can also be achieved by perturbing the underlying bipartite matching
polytope as to make it non-degenerate, see Section 6. This result can be of computational interest, since on non-degenerate polytopes the behaviour of Scarf's algorithm is univocally determined, hence no ad-hoc pivoting rule needs to be implemented. Moreover, this latter theoretical result substantiates the empirical observation that Scarf's algorithm converges fast in standard perturbations of the bipartite matching polytope [6].

Limits of Scarf's Approach: Expressing Stable Matchings. Recent work has focused on understanding the "expressive power" of algorithms for constructing stable matchings. For instance, while Gale and Shapley's algorithm [19] can be used to produce at most two stable matchings, the deferred acceptance algorithm with compensation chains can output all stable matchings [14. Biró and Fleiner [6] ask whether Scarf's algorithm can also be used to output all stable matchings of a given instance. As our next result, we show that the expressive power of Scarf's algorithm is also weak, since it will, in general, output only an exponentially small subset of stable matchings.

Our result requires $C$ to be consistent [1]. This is a common assumption that, roughly speaking, states that $C$ needs to characterize $\succ$ correctly (see Section 2.3 for a formal definition). We emphasize that all the ordinal matrices $C$ discussed in Theorems 3, [4 and extensions to the roommate setting, to hypergraphic matching [1], and to matching with couples [7], satisfy consistency. More generally, all applications of Scarf's algorithm to stable matching problems we are aware of employ a consistent matrix $C$. Now let $\operatorname{dom}(A, b, C)$ be the family of dominating vertices of $(A, b, C)$ and, for a marriage instance $\mathcal{I}$, let $\mathcal{S}(I)$ be the family of stable matchings of $\mathcal{I}$.

Theorem 5. There is a universal constant $c>1$ and, for infinitely many $n \in \mathbb{N}$, a marriage instance $\mathcal{I}_{n}$ with $n$ agents such that, for every $A, b$ induced by $\mathcal{I}_{n}$ and matrix $C$ consistent with $\mathcal{I}_{n}$, we have:

$$
\frac{\left|\mathcal{S}\left(\mathcal{I}_{n}\right)\right|}{|\operatorname{dom}(A, b, C)|}=\Omega\left(c^{n}\right) .
$$

Hence, to cover all stable matchings of a marriage instance with dominating vertices of consistent matrices, we may need exponentially many matrices. In particular, for each consistent $C$, there are exponentially many stable matchings that cannot be obtained via Scarf's algorithm, thus answering a question of [6].

### 1.3 Related Literature

As mentioned in the introduction, Scarf's lemma has been used to show the existence of objects such as cores and fractional cores [6, 37, strong fractional kernels [2, and fractional stable solutions in hypergraphs [1] and stable paths [21]. In particular, Biró and Fleiner [6] investigate Scarf's algorithm for stable matching problems and extensions, posing many intriguing question on the features of Scarf's algorithm, some of which are investigated (and answered) in this paper, see Section [1.2] For some more complex markets, many 2-stage rounding algorithms [28, [29, 30] use Scarf's algorithm as a first step to find a fractional point, which is then often rounded in a second step to a feasible or quasi-feasible solution.

To the best of our knowledge, no result on the polynomial-time convergence of Scarf's algorithm was known prior to this work. In contrast, it was known that the problem of finding a dominating vertex is PPAD-Complete, even in a restricted setting such as hypergraph matching [12, 23, 24]. PPAD [32] is a complexity class containing certain problems whose associated decision version always has a positive answer, but whose solution may be non-trivial to find. PPAD-Complete problems include the computation of Nash equilibria [11, 13] and of fixed point of Brouwer functions [32],
among others, hence the existence of a polynomial-time algorithm that finds a solution for PPADComplete problems would be surprising. Moreover, examples are known where Scarf's algorithm's path is uniquely defined and requires an exponential number of steps (independently of any complex theoretic assumption), even if the corresponding dominating vertex can be found in polynomial time (15].

A crucial component of our approach is an understanding of the feasible and ordinal bases of the bipartite matching polytope that can be visited by Scarf's algorithm. In contrast, most of the polyhedral literature on matching polytopes has focused on studying vertices and conditions for their adjacency only (see, e.g., [3, 36]) in order to, e.g., bound the diameter, or to investigate classical pivoting operations [4. Another polyhedral approach to stable matching problems studies properties of the stable marriage or roommate polytopes, i.e., the convex hull of stable matchings, focusing on topics such as their linear descriptions [17, 18, 33, 35, 41, 42] and diameters [16].

## 2 Technical Overview

### 2.1 Polynomiality of Scarf's Algorithm in the Marriage Model

Consider a marriage instance $\mathcal{I}:=(G(V, E), \succ)$, where $G$ is a bipartite graph with nodes $V=M \cup W$, and $M$ and $W$ are the set of men and women, respectively. $E=E^{\ell} \cup E^{v}$, where $E^{\ell}$ is the set of loops - one for every vertex - while $E^{v}$ is the set of valid edges (i.e., not loops). Recall that we assume that the underlying graph is complete. $|V|=\left|E^{\ell}\right|=n=2 k$ and $\left|E^{v}\right|=m=k^{2}$. For $v \in V$ and $e \in E$, we say $v$ is incident to $e$ if $v \in e . \succ$ is a preference system such that for every $v \in V$, $\succ_{v}$ strictly ranks all the edges incident to $v$ and the unique loop on $v$ such that $e \succ_{v}(v, v)$ for every $e$ incident to $v$. We let $A=\left(I \mid A^{\prime}\right)$, where $A^{\prime}$ is the incidence matrix of the graph, and the identity matrix $I$ corresponds to the loops (slack variables). We let $b=(1,1, \ldots, 1)^{T} \in \mathbb{R}^{n}$.

With $A, b$ as above, (11) defines the matching polytope of $G(V, E)$. To a fixed graph $G(V, E)$ we associate a matrix $C$ such that each dominating vertex of $(A, b, C)$ is a stable matching of $G(V, E)$, see Theorem [20. A set of columns of $A$ or $C$ (in particular, a feasible or ordinal basis) will also be interpreted as a set of edges of $G$.

We give here a high-level view of the behaviour of Scarf's algorithm on a generic iteration associated with a pair $(B, D)$ (cardinal basis and ordinal basis, respectively), and a combinatorial interpretation of intermediate vertices found on the way. We denote the valid (resp. loop) edges in $B$ by $E_{B}^{v}$ (resp. $E_{B}^{\ell}$ ), and define the graph (opti, with loops) $G_{B}=\left(V, E_{B}^{v} \cup E_{B}^{\ell}\right)$. Similarly, $G_{D}$ (resp. $E_{D}$ ) is the subgraph (resp. the subset) containing all and only the edges in $D$. The pair associated after performing a cardinal and an ordinal pivoting starting from $(B, D)$ is denoted by ( $B^{\prime}, D^{\prime}$ ).

We next introduce definitions and properties of objects associated to a given iteration. The reader can follow those in Figure 1 and Figure 2 for illustration.

1. Structure of basis $B$ : Forest with single loops. $F=\left(V, E_{B}^{v}\right)$ is a forest, and each connected component of $F$ contains exactly one edge from $E_{B}^{\ell}$ (i.e., a loop, see Lemma 21). We say therefore that $G_{B}$ has a forest with single loops structure. This fact is probably folklore, but we could not find a reference and for completeness we give a proof in Appendix A. 1 ,
2. Separator. Define $m_{k+1}=m_{1}$. Each of the ordinal bases visited by the algorithm will have a unique separator. This is an agent $m_{i} \in M$ that, among other properties, satisfies the following:
(i) the separator $m_{i}$ is incident to both the loop $e_{i}$ and some valid edge(s);
(ii) for $i^{\prime}=2, \ldots, i-1, m_{i^{\prime}}$ is incident to at least one valid edge of $G_{D}$ and no loop;
(iii) for $i^{\prime}=i+1, \ldots, k, m_{i^{\prime}}$ is only incident in $G_{D}$ to the loop $e_{i^{\prime}}$.

See Proposition 23 and Section 5.4.
3. Utility vector. Recall that the utility $u \in \mathbb{Q}^{n}$ is defined as $u_{i}=\min \left\{c_{i j}: j \in D\right\} \forall i \in[n]$. If an ordinal pivot starting from a basis with associated utility $u$ leads to a basis associated to $u^{\prime}$ with $u_{i}^{\prime}>u_{i}$ for some $i \in[n]$, we say that the utility of $i$ increases.
4. $v$-disliked edge, where $v \in M \cup W$. We say that $e_{i} \in E_{D}$ is $v_{j}$-disliked w.r.t. $D$ if $c_{i j}=u_{i}$. This defines a bijection between edges in an ordinal basis $D$ and agents, see Definition 18 ,

In each iteration, we execute the following steps:

1. Identification of man- and woman-disliked edges. We identify the set of woman(resp. man-) disliked edges as the set of edges that are $v$-disliked, for some agent $v$ that is a woman (resp. man).
2. Cardinal Pivoting: Let $\Lambda$ be the connected component of $G_{B}$ containing the separator $m_{i}$.Recall that the next basis has the form $B^{\prime}=B \cup\left\{j_{t}\right\} \backslash\left\{j_{\ell}\right\}$ and $G_{B^{\prime}}$ is a forest with single loops. We show that one of the following holds:
(a) $\Lambda$ is a connected component of $G_{B \cup\left\{e_{j_{t}}\right\}}$ containing an even cycle $C$. Then $e_{j_{\ell}}$ can be chosen to be an edge of $C$. See Figure 1
(b) In $G_{B \cup\left\{e_{j_{t}}\right\}}, \Lambda$ is joined with another connected component of $G_{B}$, as to create a component $\Gamma$ with two loops. Then $e_{j_{\ell}}$ can be chosen to be either a loop, or an edge of the path connecting the two loops. See Figure 2.

We show in Lemma 28 that if $e_{j_{t}}$ is man-disliked, we can always pick $e_{j_{\ell}}$ to be either (i) a woman-disliked edge in $D$ (Figure 11), or (ii) the loop corresponding to the separator $m_{i}$ (Figure 2).
3. Ordinal Pivoting: The new ordinal basis will be of the form $D^{\prime}=D \backslash\left\{j_{\ell}\right\} \cup\left\{j^{*}\right\}$. Recall that $j^{*}$ is uniquely determined by $D, j_{\ell}$. In Section 5.3.2, we show that if case (i) in the Cardinal Pivoting analysis holds, then $e_{j^{*}}$ is a $m_{j}$-disliked valid edge in $D^{\prime}$ for some $j \in\{2,3, \ldots, i-1\}$, see Figure 3, while if (ii) holds, then $e_{j^{*}}$ is an $m_{1}$-disliked valid edge in $D^{\prime}$ and $m_{i+1}$ is the separator in $D^{\prime}\left(\right.$ let $\left.m_{k+1}=m_{1}\right)$, see Figure 4. In both cases, $e_{j^{*}}$ is man-disliked, which provides the exact conditions for the cardinal pivot rule discussed above to apply.

We then continue the next iteration with $\left(B^{\prime}, D^{\prime}\right)$.
Our convergence analysis follows by the componentwise monotone evolution of a potential vector in $\mathbb{Z}^{2}$, see Section 5.4.

$$
\begin{equation*}
\left(i, \sum_{w \in W} u_{w}\right) \tag{3}
\end{equation*}
$$

where $i$ is the index of the current separator. That is, at each iteration one component of the potential vector strictly increases while the other does not decrease. In particular, either the new separator becomes $m_{i+1}$ and and the total utility of women (i.e., the $\ell_{1}$ norm of the subvector of $u$ restricted to women) does not decrease, or the total utility of women strictly increases, while the


Figure 1: An illustration of some of the concepts introduced for the marriage case. We let $V=\left\{m_{1}, \ldots, m_{5} ; w_{1}, \ldots, w_{5}\right\}$, and the pivoting rule when an even cycle occurs. On the left: The graph $G_{B}$, with the convention that solid edges are associated to variables from $B$ with $x$-value 1 and dotted edges to variables from $B$ with $x$-value 0 . One can see that $G_{B}$ is a forest with single loops. Edge $e_{j_{t}}=\left(m_{2}, w_{1}\right)$ will enter the basis creating an $x$-alternating cycle $Q=\left(m_{2}, w_{1}\right),\left(w_{1}, m_{3}\right),\left(m_{3}, w_{3}\right),\left(w_{3}, m_{2}\right)$. In the center: The graph $G_{D}$. All edges are full since there is no value associated to a ordinal basis. Gray arrows denote which edge is disliked by each node. It can be observed that $m_{4}$ is the separator. Edges $e_{j_{t}}$ (entering $B$ ) and $e_{j_{\ell}}$ (leaving $B$ ) are highlighted. In particular, the entering edge $e_{j_{t}}$ is $m_{2}$-disliked, and the addition of $e_{j_{t}}$ to $E_{B}$ creates a connected component with an even cycle. On the right: The cardinal pivoting removes $e_{j_{\ell}}=\left(w_{1}, m_{3}\right)$, and leads to the new feasible basis $B^{\prime}=B \backslash\left\{e_{j_{\ell}}\right\} \cup\left\{e_{j_{t}}\right\}$. Our pivoting rule indicates that, in this iteration, we can always select a woman-disliked edge inside $Q$ to leave. Notice that we may also select the leaving edge to be $\left(w_{3}, m_{2}\right)$. We arbitrarily select one when multiple choices exist.


Figure 2: An illustration of pivoting rule when a path connecting two loops occurs. On the left: The graph $G_{B}$. In the center: The graph $G_{D}$. It can be observed that $m_{4}$ is the separator. The addition of $e_{j_{t}}$ to $B$ creates a connected component with two loops. On the right: The cardinal pivoting removes $e_{j_{\ell}}$, which is the loop $\left(m_{4}, m_{4}\right)$, and leads to the new feasible basis $B^{\prime}=B \backslash\left\{j_{\ell}\right\} \cup\left\{j_{t}\right\}$.
separator is still $m_{i}$. Since both the number of separators (men) and the total utility of women are bounded by $O\left(n^{2}\right)$, the number of iterations is $O\left(n^{2}\right)$, and each iteration can clearly be performed in polynomial time.

For a combinatorial interpretation of the pivoting rule, one can think of the algorithm as adding men one by one from the queue $m_{2}, m_{3}, \ldots, m_{k}, m_{1}$. At each iteration, the separator denotes the


Figure 3: The ordinal pivot as a continuation of Figure (1) On the left: The graph $G_{D}$, where we want the $w_{1}$-disliked edge $\left(m_{3}, w_{1}\right)$ to leave. We then find the second worst choice for $w_{1}$, which is the reference edge $\left(m_{2}, w_{1}\right)$ and $m_{2}$-disliked in $D$. On the right: The graph $G_{D^{\prime}}$, where the next entering variable $e_{j^{*}}$ is a $m_{2}$-disliked valid edge. During this ordinal pivot, the utility of $w_{1}=i_{\ell}$ increases, and the utility of $m_{2}=i_{r}$ decreases, while others do not change their utilities and their disliked edges. The separator stays at $m_{4}$.


Figure 4: The ordinal pivot as a continuation of Figure 2 On the left: The graph $G_{D}$, where we want the loop ( $m_{4}, m_{4}$ ) to leave. We then find the $m_{1}$-disliked edge ( $m_{4}, w_{5}$ ) as the reference edge $e_{j_{r}}$. On the right: The graph $G_{D^{\prime}}$, where the next entering variable $e_{j^{*}}$ is a $m_{1}$-disliked valid edge. During this ordinal pivot, the utility of $m_{4}=i_{\ell}$ increases, and the utility of $m_{1}=i_{r}$ decreases, while others do not change their utilities and their disliked edges. The separator changes from $m_{4}$ to $m_{5}$.
last man introduced. One can show that we change the separator from $m_{i}$ to $m_{i+1}$ as soon as we obtain a "local" stable matching, i.e., a stable matching restricted to men $m_{2}, m_{3}, \ldots, m_{i}$ and all women. In iterations when the separator does not change, the algorithm adjusts the current matching by "improving" the matching for women (thus the increase in $\sum_{w \in W} u_{w}$ ), until a "local" stable matching is obtained.

### 2.2 Polynomial-time Convergence Through a Perturbation of the Polytope

In the previous section, we showed that a stable matching can be obtained in polynomial time by running Scarf's algorithm on the bipartite matching polytope $\mathcal{P}$ with a suitable pivoting rule. The implementation of this approach would however be non-trivial, since it needs in particular to deal
with degenerate pivots. For practical purposes, it would be desirable to run Scarf's algorithm on a perturbation of $\mathcal{P}$, since every pivot of Scarf's algorithm on non-degenerate polytopes is uniquely determined 37, hence no tailored pivoting rule needs to be implemented.

In Section 6, we show how a "classical" perturbation allows us to find a stable matching using Scarf's algorithm in polynomial time. Our approach is as follows. We perturb $\mathcal{P}$ as to construct a non-degenerate polytope $\mathcal{P}^{\epsilon}$ (see (11)). Thus, an execution of Scarf's algorithm on $\mathcal{P}^{\epsilon}$ is uniquely defined by a sequence of pairs $\left(x_{0}^{\epsilon}, D_{0}\right) \rightarrow\left(x_{1}^{\epsilon}, D_{1}\right) \rightarrow \cdots \rightarrow\left(x_{N}^{\epsilon}, D_{N}\right)$, where $x_{I}^{\epsilon}$ is a vertex of $\mathcal{P}^{\epsilon}$ and $D_{I}$ is an ordinal basis of $C$, for $0 \leq I \leq N$. Our main technical ingredient here is to show that there is a sequence of Scarf pairs $\left(B_{0}, D_{0}\right) \rightarrow\left(B_{1}, D_{1}\right) \rightarrow \cdots \rightarrow\left(B_{N}, D_{N}\right)$ satisfying our pivoting rule in the non-perturbed case (i.e., Algorithm (1) and such that, for each $I, B_{I}$ corresponds in $\mathcal{P}^{\epsilon}$ to $x_{I}^{\epsilon}$. Therefore, based on Theorem 4, we have $N=\operatorname{poly}(n)$, achieving the claimed polynomial-time convergence. The details are given in Theorem 42,

### 2.3 Limits of Scarf's Algorithm: Expressing Stable Matchings

As we argue next, under the (quite natural) consistency assumption on $C$, the set of stable matchings that Scarf's algorithm can output may be exponentially smaller than the set of all stable matchings of an instance.

Definition 6. An ordinal matrix $C$ is consistent with a marriage instance $\mathcal{I}=(G(V, E), \succ)$ with $V=\left\{v_{1}, \ldots, v_{n}\right\}$ if
(i) For any $i, \bar{\imath} \in[n]$ and $e_{j} \in E$ such that $v_{i} \in e_{j}$ and $v_{\bar{\imath}} \notin e_{j}$, we have $c_{i, j}<c_{\bar{\imath}, j}$.
(ii) For any $i \in[n]$ and $e_{j}, e_{\ell} \in E$ such that $v_{i} \in e_{j}$, $e_{\ell}$, then $c_{i j}>c_{i \ell}$ if and only if $e_{j} \succ_{v_{i}} e_{\ell}$.

In Definition 6, (i) is a regularity condition, and (ii) means that $C$ characterizes the order of $\succ$ correctly. Under the assumption that $C$ is consistent, one can deduce that every dominating vertex $x$ of $(A, b, C)$ is the characteristic vector of a stable matching of $\mathcal{I}$, where $A, b$ is induced by $\mathcal{I}$, see [1]. We emphasize that all the ordinal matrices $C$ discussed in Theorems 34] and extensions to hypergraphic matching [1] and matching with couples [7] yield consistent $C$.

We show that, when $C$ is consistent, $\operatorname{dom}(A, b, C)$ always yields a $v$-optimal stable matching, i.e., there exists an agent $v$ who is matched to her best partner among all stable matchings. We remark, in passing, that, since our implementations of Scarf's algorithm discussed in Section 2.1 and Section 2.2 are obtained with a consistent $C$, the theorem applies for those algorithms as well.

Theorem 7. Suppose $C$ is a consistent ordinal matrix and $A, b$ are as in Theorem 圂. Then any dominating vertex of $(A, b, C)$ is a characteristic vector of a v-optimal stable matching for some $v \in V$.

Hence, any "intermediate" stable matching (i.e., that does not assign any agent to their favorite stable partner) cannot be output by Scarf's algorithm because it cannot even be represented by a dominating vertex. We show in Example 51 an infinite family of instances $\mathcal{I}_{n}$ with 2 stable matchings that are $v$-optimal for some $v$, but exponentially many stable matchings, and Theorem 5 follows.

## 3 Review of Scarf's Algorithm

We discuss here some classical properties of Scarf's algorithm and introduce related definitions and notation. This section can be used by the reader as an introduction / reminder of the algorithm,
but it also presents building blocks that will be used in our arguments in future sections. Missing proofs can be found in 37.

Recall that the input to Scarf's algorithm is given by $n \times(n+m)$ nonnegative matrices $A$ and $C$ with special properties. We call $C$ an ordinal matrix (see Section 1.1).

### 3.1 Cardinal and Ordinal Pivots

To make the argument clear, we first add the standard nondegeneracy assumption that all of the variables associated with the $n$ columns of a feasible basis $B$ for the equations $A x=b$ are strictly positive (Scarf's algorithm was originally stated in this setting only [37]). Recall that at each step of Scarf's algorithm, we are given matrices $B$ and $D$, where $B$ is a $(A, b)$ basis and $D$ is an ordinal basis for $C$. The properties of $D$ imply the following.

Proposition 8. For any column $c$ of an ordinal basis $D$, there is exactly one row minimizer to be used in forming the utility vector. More formally, there is a unique row $i \in[n]$, such that $u_{i}=c_{i}$. Hence, for the other rows $\bar{\imath} \neq i$, we have $u_{\bar{\imath}}<c_{\bar{\imath}}$.

Proposition 8 gives a bijection from the $n$ columns of an ordinal basis to the $n$ rows.
Recall that an iteration of Scarf's algorithm consists of two main steps: cardinal pivot and ordinal pivot. Cardinal pivot is similar to the pivot performed by the simplex algorithm, where we have:

Lemma 9. Let $B=\left\{j_{1}, \cdots, j_{n}\right\}$ be an $(A, b)$ basis, and let $j^{\prime}$ be an arbitrary column not in $B$. Then, if (11) is nondegenerate and the feasible set $\{x \mid x \geq 0$ and $A x=b\}$ is bounded, there is a unique $j_{t} \in B$ such that $B \backslash\left\{j_{t}\right\} \cup\left\{j^{\prime}\right\}$ is an $(A, b)$ basis.

The previous is a standard result in linear programming, which says we can arbitrarily choose an outside column to enter the basis while a unique column leaves. A symmetric property holds for the ordinal pivot. An arbitrary column in an ordinal basis visited by the algorithm may be removed and a unique column introduced from outside so that the new set of columns is also an ordinal basis.

Lemma 10. Let $D=\left\{j_{1}, \cdots, j_{n}\right\}$ be an ordinal basis of $C$ and $\ell \in[n]$. Then there exists a unique column $j^{*} \notin D$ such that $D \cup\left\{j^{*}\right\} \backslash\left\{j_{\ell}\right\}$ is an ordinal basis of $C$.

Recall that the procedure to find such a new column $j^{*}$ is called ordinal pivot. We give the formal definition of ordinal pivot as follows:

Definition 11 (Ordinal pivot). Consider an ordinal basis $D$ and a specific column $j_{\ell}$ to be removed from it. In the $n \times(n-1)$ matrix of remaining columns, define

$$
\bar{u}_{i}=\min _{j \in D \backslash\{j \ell\}} c_{i, j} .
$$

There exists exactly one column $j_{r}$ that contains two row minimizers in forming $\bar{u}$ (according to Proposition 8). We call $j_{r}$ the reference column. Between the two minimizers, one of them is new and the other is a row minimizer of the original ordinal basis. Let the row associated with the former have an index $i_{\ell}$ (i.e., $c_{i_{\ell}, j_{\ell}}=u_{i_{\ell}}<\bar{u}_{i_{\ell}}=c_{i_{\ell}, j_{r}}$ ) and the latter have an index $i_{r}$ (i.e., $\left.c_{i_{r}, j_{r}}=u_{i_{r}}=\bar{u}_{i_{r}}\right)$. Let $K$ denote the columns in $C$ such that $k \in K$ if

$$
\begin{equation*}
c_{i, k}>\bar{u}_{i}, \text { for all } i \neq i_{r} . \tag{4}
\end{equation*}
$$

Of the columns in $K$, select the one which maximizes $c_{i_{r}, k}$, i.e.

$$
j^{*}=\underset{k \in K}{\arg \max } c_{i_{r}, k} .
$$

An ordinal pivot step introduces this column $j^{*}$ into the ordinal basis, as to form the new ordinal basis $D^{\prime}=D \backslash\{j \ell\} \cup\left\{j^{*}\right\}$.

It can be shown that $j=j^{*}, j_{\ell}$ are the only two columns that make $D \backslash\{j \ell\} \cup\{j\}$ an ordinal basis. This fact tells us that the number of ordinal bases which contains any given $n-1$ columns can only be 0 or 2 . It also suggests that the ordinal pivots are "reversible": If $j_{\ell}$ is eliminated from a basis and $j^{*}$ brought in, then $j^{*}$ may be eliminated from the new basis and the original basis will be obtained.

It is useful to analyze the change of utility vector in an ordinal pivot. We follow the notation from Definition 11 .

Lemma 12. Consider the two utility vectors $u$ and $u^{\prime}$ associated to $D, D^{\prime}$, respectively. Then, when going from $D$ to $D^{\prime}$, the utility of $i_{\ell}$ increases while the utility of $i_{r}$ decreases, and others are indifferent. Formally,

$$
\begin{gathered}
u_{i_{\ell}}^{\prime}=c_{i_{\ell}, j_{r}}>c_{i_{\ell}, j_{\ell}}=u_{i_{\ell}}, \\
u_{i_{r}}^{\prime}=c_{i_{r}, j^{*}}<c_{i_{r}, j_{r}}=u_{i_{r}}, \\
u_{i}^{\prime}=u_{i}, \text { for } i \in[n], i \neq i_{\ell}, i_{r} .
\end{gathered}
$$

### 3.2 Scarf Pairs, Almost-feasible Ordinal Bases, and the Termination Condition for the Algorithm

We have already discussed in Section 1.1 how the sequence of ordinal and cardinal pivoting leads, in finite time, to a dominating vertex. We next give more details on the (ordinal) bases visited and the termination condition of the algorithm.

Definition 13 (Scarf pair). Suppose Scarf's algorithm starts with initial feasible basis $B_{0}$ and ordinal basis $D_{0}$. For $i \geq 1$, if $B_{i-1} \neq D_{i-1}$, then the $i$-th iteration consists first of a cardinal pivot $\left(B_{i-1}, D_{i-1}\right) \rightarrow\left(B_{i}, D_{i-1}\right)$ and then, if $B_{i} \neq D_{i-1}$, of an ordinal pivot $\left(B_{i}, D_{i-1}\right) \rightarrow\left(B_{i}, D_{i}\right)$, where the pivots are defined in the previous section. Let

$$
B_{0}, B_{1}, \ldots, B_{I} \quad \text { and } \quad D_{0}, D_{1}, \ldots, D_{I}
$$

be the sequence of feasible (resp. ordinal) bases visited by Scarf's algorithm such that $B_{I}=D_{I}$ when it terminates. For $0 \leq i \leq I-1$, we call $\left(B_{i}, D_{i}\right)$ a Scarf pair.

Hence, an iteration always starts with a Scarf pair and ends up with a new Scarf pair if the algorithm does not terminate. By Lemma 9 and Lemma 10, any Scarf pair ( $B_{i}, D_{i}$ ) has $\left|B_{i} \cap D_{i}\right|=$ $n-1$.

Lemma 14. Throughout the algorithm, $B_{i-1} \neq D_{i-1}$ and $B_{i}=D_{i}$ if and only if in the $i$-th iteration column 1 leaves the basis $B_{i-1}$ or column 1 is introduced in the ordinal basis $D_{i-1}$. One of these two occurrences happen after a finite number of iterations, i.e., $I<\infty$ in Definition 13. In particular, a dominating vertex exists.

We next present properties shared by ordinal bases visited by our algorithm.

Definition 15 (Almost-feasible ordinal bases). An ordinal basis $D=\left\{j_{1}, \ldots, j_{n}\right\}$ is almost-feasible if $1 \notin D$, and there exists a feasible basis $B$ such that $B=\left\{1, j_{1}, \ldots, j_{t-1}, j_{t+1}, \ldots, j_{n}\right\}$ for some $t \in[n]$. That is, $1 \in B$ and $R=B \cap D$ has cardinality $n-1$. We say that $B$ is associated to $D$ and call the set $R$ remaining columns, and let $A_{R}, D_{R}$ denote the submatrices of $A, C$ corresponding to the index set $R$, respectively.

By Lemma 14. Scarf's algorithm terminates if column 1 enters the ordinal basis $D$ or leaves the feasible basis $B$. Moreover, throughout the algorithm, for each Scarf pair $(B, D)$, by construction $B$ and $D$ differ in at most one entry. Hence, we deduce the following.

Lemma 16. Let $D$ be an ordinal basis visited by some execution of Scarf's algorithm on the input from Theorem 园. Then either $D$ is the final basis visited by the algorithm, or $D$ is almost-feasible.

### 3.3 Dealing with Degeneracy

Notice that in the framework above, the behaviour of Scarf's algorithm is uniquely determined by the input. In contrast, when Scarf's algorithm is applied to a degenerate polytope, i.e., such that there are strictly more than $n$ constraints that are tight at some vertex, then many options may be possible for cardinal pivoting. Hence, the output of Scarf's algorithm is not unique and depends on the cardinal pivoting rule. Moreover, cycling may happen. However, definitions and properties from Section 3.2 apply, with the exception of finite convergence, which is not guaranteed.

Lemma 17. $B_{i-1} \neq D_{i-1}$ and $B_{i}=D_{i}$ if and only if in the $i$-th iteration column 1 leaves the basis $B_{i-1}$ or column 1 is introduced in the ordinal basis $D_{i-1}$.

In Section 5, we deal with matching polytope that are highly degenerate, and we show that with our cardinal pivoting rules Scarf's algorithm converges, and does so in polynomial time.

There is however another standard way to deal with degeneracy: perturbation (this is discussed for the marriage case in Section (6). Indeed, perturb the right hand side vector to $b^{\prime}$ at the beginning to make the polytope nondegenerate. Then, Scarf's algorithm outputs a dominating basis $B$ w.r.t. $\left(A, b^{\prime}, C\right)$. This implies that $B$ is an ordinal basis for $C$. If $B$ is also an $(A, b)$ basis, then $B$ is an $(A, b)$ basis [5], then it is dominating for $(A, b, C)$. Therefore, if the perturbation is small enough so that every $(A, b)$ is also an $\left(A, b^{\prime}\right)$ basis, then the output basis corresponds to a dominating vertex of the original polytope.

## 4 Additional Facts and Notations

We now discuss some facts and notation that are used throughout the rest of the paper.

### 4.1 Graphs, Edges, Matching

Recall that we start with the input $\mathcal{I}=(G(V, E), \succ)$, where $G$ is a graph with a set $V$ of $n$ nodes, $E$ is the set of edges given by the (disjoint) union of $n$ loops $E^{\ell}$ and $m$ valid edges $E^{v}$. For $v \in V$ and $e \in E$, we say $v$ is incident to $e$ if $v \in e . \succ$ is a preference system such that for every $v \in V$, $\succ_{v}$ ranks all the edges incident to $v$ and the unique loop on $v$ such that $e \succ_{v}(v, v)$ for every $e \in E^{v}$ incident to $v$. When $e=(v, u), e^{\prime}=\left(v, u^{\prime}\right)$, we also write $u \succ_{v} u^{\prime}$ when $e \succ_{v} e^{\prime}$.

For a graph $G(V, E)$, define the degree of a node $v \in V$ as the number of edges in $G$ (including loops) incident to $v$, and denote it by $\operatorname{deg}_{G}(v)$. Note that, for the same node $v \in V$, the degree is subject to $G$. For example, $\operatorname{deg}_{G_{B}}(v), \operatorname{deg}_{G_{D}}(v)$ may be different.

We can then redefine the concept of matching. We say that an edge $e$ is incident to a node $v$ if the latter is an endpoint of the former. A matching is an edge set $\mu$ such that for each node $v$, there is exactly one edge $e \in \mu$ incident to $v$. Equivalently, $\mu \subset E$ is a matching if the subgraph $G_{\mu}=(V, \mu)$ is such that every node $v \in V$ has $\operatorname{deg}_{G_{\mu}}(v)=1$. We say $\mu$ properly matches $v$ if a valid edge in $\mu$ is incident to $v$. Hence, if a matching $\mu$ does not properly match node $v$, then the loop $e \in \mu$, which implies $v$ is unmatched in the classical sense.

### 4.2 Bases and Related Objects

Because of the structure of our input, the matrix $A$ (resp. $C$ ) has exactly one row per agent, and exactly one column per edge, including loops. Therefore, we sometime overload notation and use the same symbol to denote an agent/edge and a row/column, with the exact meaning being always clear from the context. More precisely:

- $m, w, v$ (possibly with subscripts/superscripts) are used to refer to agents (with $m \in M$, $w \in W$ and $v \in V)$ or to the corresponding node or row index in either the $A$ or $C$ matrix.
- $(m, m),(w, w),(m, w),\left(v_{i}, v_{\ell}\right)$ denote edges or the corresponding column indices of $A$ or $C$. Note that we adopt the ordered pair notation even though all graphs are undirected. This is motivated by that fact that the ordered pair allows us to distinguish between men (first entry) and woman (second entry). We sometime also denote an edge by $e_{j}$, with $j \leq n$ for loops and $j>n$ for valid edges. Accordingly, $a_{j}, c_{j} \in \mathbb{Q}^{n}$ are the column vectors corresponding to $e_{j}$ in matrix $A, C$, respectively.

We similarly overload notation for bases / ordinal bases as follows:

- $B$ is some feasible basis for $A$. It can be an index set of $n$ columns, or of $n$ edges of $G$ (including loops). $A_{B}$ is the submatrix obtained from $A$ by restricting to columns of $B . G_{B}=\left(V, E_{B}\right)$ is a subgraph of $G$ only maintaining the edges in $B$.
- $D$ is some ordinal basis of $C$. It can be an index set of $n$ columns, or of $n$ edges of $G$ (including loops). $C_{D}$ is the submatrix obtained from $C$ by restricting to columns of $D . G_{D}=\left(V, E_{D}\right)$ is a subgraph of $G$ only maintaining the edges in $D$.

Other relevant notation includes vectors associated to bases:

- $x \in \mathbb{R}^{n+m}$ is a feasible solution of the desired polytope. It assigns weights on every edge on $E$. The $x$-value of an edge, $x_{e}$, is the value of the component of $x$ corresponding to the edge $e$.
- For a feasible basis $B$, let $x$ be the basic feasible solution associated with $B$. Define $\mu_{B}$ as the matching with edges given by $\operatorname{supp}(x)$, i.e., $e \in \mu_{B}$ iff $x_{e}>0$.
- For an ordinal basis $D$, let $u_{D} \in \mathbb{Z}^{n}$ denote the utility vector associated with $D$, i.e., $\left(u_{D}\right)_{i}=$ $\min _{j \in D} c_{i j}$. If there is no ambiguity, we omit the subscript $D$ and just denote by $u$ the utility vector of $D$.


### 4.3 Paths, Cycles

We redefine the concept of path in graphs as follow. Let $F \subset[n+m]$ be a subset of columns and $A_{F}$ be the submatrix of $A$ restricted to columns in $F$. Consider the graph $G_{F}=\left(V, E_{F}\right)$ where $E_{F}$ is the edge set corresponding to columns in $F$. A path $P$ in $G_{F}$ is defined as a sequence of edges

$$
P=\left(v_{1}, v_{2}\right),\left(v_{2}, v_{3}\right), \ldots,\left(v_{\ell-1}, v_{\ell}\right)
$$

with $v_{1} \neq v_{\ell}$, such that the two nodes in some parentheses can coincide, no pair is repeated, and the valid edges in $P$ form a path in the classical sense. Hence, paths may contain loops. A cycle $Q$ is defined as a cycle in the classical sense. We denote by $V_{P}$ and $E_{P}$ the vertices and edges of a path, respectively. A similar notation is employed for $Q$. Sometimes, in order to stress its vertices, when denoting a path $P$ (resp. a cycle $Q$ ) we also add between pairs of consecutive edges the vertex they share.

Let $x \in \mathbb{R}^{n+m}$. We say $P$ (resp., $Q$ ) is $x$-alternating if the $x$-value of edges on $P$ (resp., $Q$ ) alternate between 0 and 1 . We say $P$ is $x$-augmenting if it is $x$-alternating and the first edge is a loop with $x$-value 1, i.e., $v_{1}=v_{2}$, and $x_{\left(v_{1}, v_{1}\right)}=1$. An illustration of the concepts of path and cycle is given in Figure 5.

## $4.4 \quad v$-disliked Edges

Thanks to the additional properties of Scarf's algorithm discussed in Section 3, we can now formalize the concept of disliked edge first mentioned in Section 2.1,

Definition 18 ( $v$-disliked Edge). For an ordinal basis $D$ and $i \in[n]$, let $j \in D$ be such that $u_{i}=c_{i j}$, where the existence and uniqueness of $j$ follows from Proposition 88. We call $e_{j}$ the $v_{i}$-disliked edge (in D).

In other words, there is a bijection between the rows (nodes) and columns (edges) given by $D$, and we denote this bijection by saying that $v_{i}$ dislikes $e_{j}$. The term "dislikes" comes from the fact that, if one interprets entries in $C$ as $v_{i}$ 's evaluation of all edges of $G$ (including those not incident to $v_{i}$ ), then the $v_{i}$-disliked edge will be $e_{\bar{\jmath}}$ for the unique minimizer $\bar{\jmath}$ of $c_{i j}$ over all $j \in D$ - hence the edge achieving the worst evaluation according to $v_{i}$. An important observation is that the bijection is subject to $D$ : When we say $v_{i}$ dislikes $e_{j}$, we need to specify which $D$ is referred to - but if there is no ambiguity, we omit $D$.

## 5 Polynomiality of Scarf's Algorithm in the Marriage Model

### 5.1 The Bipartite Matching Polytope and an Ordinal Matrix Associated to a Marriage Instance

We introduce here the ordinal matrix $C$ used in Theorem 4 and its features. Interestingly, $C$ is a special case of the matrix defined by [7].

Consider a complete instance $\mathcal{I}$ with $k$ men and $k$ women, denoted respectively be $M=$ $\left\{m_{1}, \ldots, m_{k}\right\}$ and $W=\left\{w_{1}, \ldots, w_{k}\right\}$ and arranged in a complete graph $G$. For $i \in[k]$, we also denote $m_{i}$ by $v_{i}$ and $w_{i}$ by $v_{k+i}$. Recall $n=2 k, m=2 k+k^{2}$. For edges, we let $e_{1}, \ldots, e_{n}$ denote the loops containing nodes $v_{1}, \ldots, v_{n}$ and for $i \in[n], e_{k(i+1)+1}, e_{k(i+1)+2}, \ldots, e_{k(i+1)+k}$ denote the edges that connect $m_{i}$ and women following the decreasing order of $m_{i}$ 's preference.


Figure 5: An illustration of the concepts of path and cycle, with the convention that solid edges (resp. dotted edges) are associated to variables from $F$ with $x$-value 1 (resp. $x$ value 0$) . \quad P=\left(v_{1}, v_{1}\right),\left(v_{1}, v_{2}\right),\left(v_{2}, v_{3}\right),\left(v_{3}, v_{4}\right),\left(v_{4}, v_{7}\right),\left(v_{7}, v_{8}\right),\left(v_{8}, v_{9}\right),\left(v_{9}, v_{10}\right),\left(v_{10}, v_{10}\right)$ is an $x$-augmenting path, and any sequential subset of $P$ is an $x$-alternating path. $Q=$ $\left(v_{1}, v_{2}\right),\left(v_{2}, v_{3}\right),\left(v_{3}, v_{4}\right),\left(v_{4}, v_{5}\right),\left(v_{5}, v_{6}\right),\left(v_{6}, v_{1}\right)$ is a cycle. $Q$ is not $x$-alternating since there are consecutive 0 's such as $\left(v_{3}, v_{4}\right),\left(v_{4}, v_{5}\right)$.

Hence, with the definition of edge given in Section 4.1 (1) defines the matching polytope of $G$ when the entry $(i, j)$ of the $A$ matrix is given as follows:

$$
a_{i j}=\left\{\begin{array}{lc}
1, & \text { if } e_{j} \text { is incident to } v_{i} \\
0, & \text { others }
\end{array} .\right.
$$

For the construction of $C$, we define

$$
\mathcal{S}=\{0\}, \mathcal{M}=\{1, \ldots, k\}, \mathcal{L}=\left\{k+1, \ldots, k^{2}\right\}, \mathcal{X} \mathcal{L}=\left\{k^{2}+1, \ldots, k^{2}+2 k-1\right\} .
$$

For any row $i$, assign

$$
c_{i j}=k+1-\ell \text {, if } a_{i j}=1 \text { and } e_{j} \text { is the } \ell \text {-th best candidate for node } v_{i} .
$$

This implies $c_{i i}=0 \in \mathcal{S}$, and $c_{i j} \in \mathcal{M}$ if $e_{j}$ is a valid edge incident to $v_{i}$. All entries $c_{i j}$ where $j$ does not correspond to edges incident to $i$ have values in $\mathcal{L} \cup \mathcal{X} \mathcal{L}$, as discussed below.

If $e_{j}$ is a valid edge but not incident to $v_{i}$, then assign $c_{i j}$ a number in $\mathcal{L}$. We have $k(k-1)$ such edges, matching the cardinality of $\mathcal{L}$. We distribute the numbers in $\mathcal{L}$ according to decreasing order from left to right. In other words, if $e_{j}, e_{\ell}$ are not incident to $v_{i}$ and $j, \ell>2 k$, then $c_{i j}, c_{i \ell} \in \mathcal{L}$ and $c_{i j}>c_{i \ell}$ if and only if $j<\ell$.

If $e_{j}$ is a loop, but $j \neq i$, we assign $c_{i j}$ a number in $\mathcal{X} \mathcal{L}$. Similarly, we distribute the numbers in $\mathcal{X} \mathcal{L}$ according to a decreasing order from left to right on those vacant positions. Note that $C$ is a consistent ordinal matrix, according to the definitions given in Section 1.1 and Section 2.3

Example 19. Given the instance with $k=2$ and the following preference lists:

$$
m_{1}: w_{2} \succ w_{1}, \quad m_{2}: w_{1} \succ w_{2}, \quad w_{1}: m_{2} \succ m_{1}, \quad w_{2}: m_{1} \succ m_{2}
$$

we can construct the $C$ matrix:

$$
\left.\begin{array}{c} 
\\
m_{1} \\
m_{2} \\
w_{1} \\
w_{2}
\end{array} \begin{array}{cccccccc}
m_{1} & m_{2} & w_{1} & w_{2} & \left(m_{1}, w_{2}\right) & \left(m_{1}, w_{1}\right) & \left(m_{2}, w_{1}\right) & \left(m_{2}, w_{2}\right) \\
0 & 7 & 6 & 5 & 2 & 1 & 4 & 3 \\
7 & 0 & 6 & 5 & 4 & 3 & 2 & 1 \\
7 & 6 & 0 & 5 & 4 & 1 & 2 & 3 \\
7 & 6 & 5 & 0 & 2 & 4 & 3 & 1
\end{array}\right) .
$$

In this instance, $\mathcal{L}=\{3,4\}, \mathcal{X} \mathcal{L}=\{5,6,7\}$, and they are assigned in decreasing order from left to righht on each row.

The way we distribute the numbers in $\mathcal{L}, \mathcal{X} \mathcal{L}$ is irrelevant to the next theorem, so there are multiple $C$ 's for which the following result is valid. However, we set them as discussed above in order to: (i) satisfy the ordinal matrix conditions (cf. Section 1.1); (ii) provide a clear structure of ordinal bases, on which we will build upon in the next section. Since matrix $C$ is a special case of the matrix from Theorem 3 from [7, we conclude the following.

Theorem 20 ([7). Any dominating basis for (A,b,C) defined above corresponds to a stable matching of $\mathcal{I}$.

The design and analysis of the algorithm follow the high-level view of the polynomial-time pivoting rule given in Section 2.1 and are organized as follows. In Section 5.2.1 we introduce definitions and show the forest with single loops structure of basis of the matching polytope. In Section 5.2.2 we discuss properties that ordinal basis will have throughout the algorithm, which allow us to define the separator and investigate the disliked relation. Features of cardinal and ordinal pivots are presented in Section 5.3.1 and Section 5.3.2, respectively. We show the convergence of the algorithm in Section 5.4. In Section 6. we show that polynomial-time convergence can also be proved if we apply Scarf's algorithm to a suitably non-degenerate perturbation of the bipartite matching polytope (recall that, in a non-degenerate polytope, the behaviour of Scarf's algorithm is uniquely determined).

### 5.2 Basic Properties

### 5.2.1 Structure of $B$ : Forest with Single Loops

Next two lemmas characterize the bases of the matching polytope. The proofs are fairly standard (see, e.g., the characterization of bases in network polytopes [5) and given for completeness in Appendix A. 1 .

Lemma 21. Let $B$ be a set of $n$ columns of $A$ and $E_{B}=E_{B}^{v} \cup E_{B}^{\ell}$ be the set of corresponding edges of $G$, where $E_{B}^{v}$ consists of all valid edges and $E_{B}^{\ell}$ consists of all loops in $E_{B} . B$ is a (possibly infeasible) basis of $A$ if and only if

1. $\left(V, E_{B}^{v}\right)$ is a forest.
2. Each connected component of the forest has exactly one loop in $E_{B}^{\ell}$.

Lemma 22. Consider any basis $B$ of $A$. Up to permuting rows and columns, the submatrix $A_{B}$ of $A$ corresponding to $B$ has the form

$$
A_{B}=\left(\begin{array}{cccc}
B_{1} & & & \\
& B_{2} & & \\
& & \ddots & \\
& & & B_{\tau}
\end{array}\right) \quad \text { where } B_{\omega}=\left(\begin{array}{cccc}
1 & * & * & * \\
& 1 & \ddots & * \\
& & \ddots & * \\
& & & 1
\end{array}\right)
$$

i.e., $B_{\omega}$ is an upper-right matrix with $1 s$ on diagonal, for $\omega \in[\tau]$ (in the matrices above, no entry means 0 , and $*$ means either a 0 or 1 ). Moreover, in each matrix $B_{\omega}$, the first column corresponds to a loop, and each other column has exactly two entries equal to 1 .

The structure presented in Lemma 21 is called forest with single loops. An illustration is given in Figure 1 .

### 5.2.2 Structure of $D$ : Almost-feasibility and Separator

Recall that all ordinal bases visited by the algorithm are almost-feasible, and $|D \cap B|=n-1$ for some feasible basis $B$ such that $(B, D)$ is a Scarf pair (see Section 3.2 and Section 3.3). Also recall that for an almost-feasible basis $D$, we let be $E_{D}$ the set of edges corresponding to columns in $D$ (including loops). Note that, if $D$ is an almost-feasible ordinal basis, then there is a submatrix of $C_{D}$ of size $n \times(n-1)$ that has the structure from Lemma 22, minus the first column of $B_{1}$ (which corresponds to the loop incident to node $m_{1}$, e.g., column 1).

This section is devoted to the proof of the following proposition, containing properties of almostfeasible ordinal bases.

Proposition 23. Let $D$ be an almost-feasible ordinal basis and $u$ the utility vector associated to it. If $u_{1} \in \mathcal{L}$ and $D$ contains at least one element in $\{2, \ldots, k\}$, then there exists an index $i, 2 \leq i \leq k$, such that in the graph $G_{D}=\left(V, E_{D}\right)$ :

1. $m_{i}$ is incident to both a loop and one or more valid edges.
2. $m_{2}, \ldots, m_{i-1}$ are incident to valid edges only; $m_{i+1}, \ldots, m_{k}$ are incident to loops only.
3. $m_{1}$ is not incident to any edge.

Following the discussion in Section 2.1, we call $m_{i}$ the separator of $D$.
An illustration of Proposition 23 is given in Figure [1 We remark that Proposition 23 and the related definition of separator apply only as long as $u_{1} \in \mathcal{L}$. As we will see in Section 5.4, we will define $m_{1}$ to be the separator when $u_{1} \in \mathcal{M}$.

It is useful to remark that we will frequently use the relation $B=R \cup\{1\}, D=R \cup\left\{j_{t}\right\}$. We start with some properties of the utility vector of an almost-feasible-basis.

Lemma 24. The utility vector of an almost-feasible ordinal basis $D$ satisfies

$$
u_{i} \in \mathcal{S} \cup \mathcal{M}, \forall i \neq 1
$$

Proof. Since $D$ has $n-1$ columns in common with a feasible basis $B$, and does not contain the first column, we have $B \backslash\{1\} \subset D$. By Lemma 22, any submatrix $A_{B}$ associated to a feasible basis $B$ has a special diagonal form. In particular, for each row $i \neq 1$, an edge incident to $v_{i}$ (possibly, a loop) appears among the columns of $R$. Hence, for every row $i$, except possibly the first, there is a column $j \in R$ such that $c_{i, j} \in \mathcal{S} \cup \mathcal{M}$. Thus the minimum element of those rows in $D$ must belong to $\mathcal{S} \cup \mathcal{M}$, which completes the proof.

By Definition 18, we can find a one-to-one correspondence of rows and columns in $D$. Recall that, intuitively, for each node $v_{i} \neq v_{1}$, the minimizer of row $i$ in $D$ corresponds to the worst choice between all edges incident to $v_{i}$ among the columns in $D$.

Clearly, if a loop $e_{i}$ satisfies $i \in D$, then $e_{i}$ is $v_{i}$-disliked since $c_{i, i}=0$ is the minimizer. Now consider a valid edge $e_{j}=(m, w)$. Column $c_{j}$ contains numbers in $\mathcal{L}$, except in rows $m$ and $w$. If $e_{j} \in E_{D}$, then as a corollary of Lemma [24, only $m, w$ or $m_{1}$ can dislike $e_{j}$ :

Corollary 25. Let $D$ be an almost-feasible basis. For any valid edge $e_{j}=(m, w)$, if $e_{j} \in E_{D}$, then $e_{j}$ can only be m-disliked, w-disliked, or $m_{1}$-disliked, and it is exactly one of the three.

Lemma 26. Let $D$ be almost-feasible basis and $B$ a feasible basis associated to it. If $u_{1} \in \mathcal{L}$, then there exists $\bar{w} \in W$ such that $(\bar{w}, \bar{w}) \in E_{B} \cap E_{D}$ and $\bar{w}$ is not properly matched in $\mu_{B}$.
Proof. Let $B$ be the associated feasible basis such that $D=B \cup\left\{j_{t}\right\} \backslash\{1\}$ as in Definition 15. Then $R \subset D . B$ corresponds to a feasible matching $\mu_{B}$. Since $u_{1} \in \mathcal{L}$, no edge is incident to $m_{1}$ in $E_{D}$, thus no valid edge is incident to $m_{1}$ in $E_{B}$, which implies $m_{1}$ is not properly matched in $\mu_{B}$. Since $|M|=|W|$, there exists some woman $\bar{w}$ who is also not properly matched. Thus $x_{(\bar{w}, \bar{w})}=1$, i.e., the loop $(\bar{w}, \bar{w}) \in E_{B} \cap E_{D}$.

Next lemma shows a fundamental property en route to the proof of Proposition 23.
Lemma 27. Consider any almost-feasible ordinal basis $D$, and suppose $u_{1} \in \mathcal{L}$. Then:
(i) If $i \notin D(2 \leq i \leq k)$ and $\ell \leq i$, then $\ell \notin D$.
(ii) If there exists $2 \leq i \leq k$ such that $i-1 \notin D$ and $i \in D$, then the rightmost column in $D$ corresponds to a valid edge ( $m_{i}, w$ ) with some $w \in W$.
Proof. (i) Since $D$ is almost-feasible, we always have $1 \notin D$. The case $i=2$ is trivial. Consider $3 \leq i \leq k$ and $i \notin D$. Assume by contradiction that there exists $2 \leq \ell<i$ such that $\ell \in D$.

Let $\bar{w}$ be the woman not properly matched whose existence is guaranteed by Lemma 26, Consider the edge $\left(m_{\ell}, \bar{w}\right)$. We show $\left(m_{\ell}, \bar{w}\right) \notin E_{D}$. Assume by contradiction $\left(m_{\ell}, \bar{w}\right) \in E_{D}$. Then $\left(m_{\ell}, \bar{w}\right)$ is neither $m_{\ell^{-}}$-disliked nor $\bar{w}$-disliked, since $\left(m_{\ell}, m_{\ell}\right),(\bar{w}, \bar{w}) \in E_{D}$. By Corollary 25, $\left(m_{\ell}, \bar{w}\right)$ is $m_{1^{-}}$ disliked. ( $m_{\ell}, \bar{w}$ ) is the rightmost column in $D$, since by definition of $C$, any column $j$ to the right of $\left(m_{\ell}, \bar{w}\right)$ satisfies $c_{1 j}<c_{1\left(m_{\ell}, \bar{w}\right)}$, contradicting the fact that $u_{1}=c_{1\left(m_{\ell}, \bar{w}\right)}$. Using the same argument, since any valid edge incident to $m_{i}$ corresponds to a column on the right of ( $m_{\ell}, \bar{w}$ ) (recall $i>\ell$ ), no valid edge in $E_{D}$ is incident to $m_{i}$. Moreover, we know $\left(m_{i}, m_{i}\right) \notin E_{D}$ by hypothesis. Hence, no edge is incident to $m_{i}$ in $B$, either, a contradiction.

Hence, $\left(m_{\ell}, \bar{w}\right) \notin E_{D}$. Since $D$ is an ordinal basis, there exists $a$ such that $u_{a}>c_{a,\left(m_{\ell}, \bar{w}\right)}$. Notice that column $\left(m_{\ell}, \bar{w}\right)$ has entries in $\mathcal{L}$, except for rows $m_{\ell}, \bar{w}$, that have entries in $\mathcal{M}$. Hence, by Lemma 24, we have $a \in\left\{m_{1}, m_{\ell}, \bar{w}\right\}$. Since $\left(m_{\ell}, m_{\ell}\right),(\bar{w}, \bar{w}) \in E_{D}$, we have $u_{\left(m_{\ell}, m_{\ell}\right)}=u_{(\bar{w}, \bar{w})}=0$. Hence, $a=m_{1}$. Recall from above that $u_{1}$ is realized at the rightmost column of $D$. Since, by hypothesis $i \notin D$ and $\ell<i$ and by construction all edges incident to node $m_{i}$ follow all columns from $D$, we have $u_{1} \leq c_{\left(m_{i}, \mu_{B}\left(m_{i}\right)\right)}<c_{\left(m_{\ell}, \bar{w}\right)}$, obtaining the required contradiction.
(ii) By (i), $1, \ldots, i-1 \notin D$ and $i, i+1, \ldots, k \in D$. Now consider the $m_{1}$-disliked edge $e=\left(m_{\ell}, w\right)$ in $D$. Since $u_{1} \in \mathcal{L}, e$ is not incident to $m_{1}$ and it is therefore the rightmost column in $D$.

If $\ell \leq i-1$, then $e$ is also $m_{\ell}$-disliked because when $\ell \notin D$, the unique $m_{\ell}$-disliked edge occurs at the rightmost entry from the set of columns incident to $m_{\ell}$, which is exactly $e$. Since $m_{\ell} \neq m_{1}$, we contradict the bijection from Definition 18 ,

If $\ell \geq i+1$, then consider the edge $\left(m_{i}, \bar{w}\right)$. If $\left(m_{i}, \bar{w}\right) \in E_{D}$ then neither $m_{i}$ (since $\left(m_{i}, m_{i}\right) \in$ $E_{D}$ ), nor $\bar{w}$ (since $\left.(\bar{w}, \bar{w}) \in E_{D}\right)$, nor $m_{1}$ (since $e \in E_{D}$ ) dislike ( $\left.m_{i}, \bar{w}\right)$. This contradicts Corollary 25. If $\left(m_{\ell}, \bar{w}\right) \notin E_{D}$, then the corresponding column $c_{\left(m_{\ell}, \bar{w}\right)}$ is strictly greater than $u$. Both are contradictions.

Therefore $\ell=i$ and $e=\left(m_{i}, w\right)$ is the rightmost column for some $w \in W$.
We now prove Proposition [23, Consider the graph $G_{D}=\left(V, E_{D}\right)$. Let $2 \leq i \leq k$ such that $i-1 \notin D$ and $i \in D$. Since $u_{1} \in \mathcal{L}, m_{1}$ is not incident to any edge, proving 3. $i \in D$ implies that $m_{i}$ is incident to a loop, while Lemma 27, part (ii) implies that $m_{i}$ is incident in $G_{D}$ to one valid edge $\left(m_{i}, w\right)$. This proves 1. By Lemma 27, part (i), $m_{2}, \ldots, m_{i-1}$, are not incident to loops, but each of them must be incident to at least one valid edge by the definition of almost-feasibility and Lemma [22. On the other hand, $m_{i+1}, \ldots, m_{k}$ are only incident to loops, by Lemma 27(i). This shows 2 and concludes the proof of Proposition 23.

### 5.3 Pivoting

In the next two sections, we will discuss how Scarf's algorithm performs a generic iteration, as long as $u_{1} \in \mathcal{L}$. As we will see later, the behavior of the algorithm is also similar when $u_{1} \in \mathcal{M}$, but the arguments are slightly different.

Suppose we have a Scarf pair $(B, D)$, with $x$ being the basic feasible solution associated to $B$ and $u$ the utility vector associated to $D$. We let

$$
\begin{equation*}
B=\left\{1, j_{1}, \ldots, j_{t-1}, j_{t+1}, \ldots, j_{n}\right\}, D=\left\{j_{1}, \ldots, j_{t}, \ldots, j_{n}\right\}, R=B \cap D=B \backslash\{1\} . \tag{5}
\end{equation*}
$$

Following Proposition [23, we let $m_{i}$ be the current separator. Recall that an iteration starts with a cardinal pivot such that $j_{t}$ enters $B$ and some $j_{\ell} \neq j_{t}$ leaves $B$, leading to a new feasible basis $B^{\prime}=\left\{1, j_{1}, \ldots, j_{\ell-1}, j_{\ell+1}, \ldots, j_{n}\right\}$ associated to the basic feasible solution $x^{\prime}$. If $j_{\ell} \neq 1$, this iteration continues with an ordinal pivot such that $j_{\ell}$ leaves $D$ and some $j^{*} \neq j_{\ell}$ enters $D$. We have therefore a new ordinal basis $D^{\prime}=\left\{j^{*}, j_{1}, \ldots, j_{\ell-1}, j_{\ell+1}, \ldots, j_{n}\right\}$, to which a new utility vector $u^{\prime}$ is associated.

If $j^{*} \neq 1$, one iteration ends and we continue to the next.

### 5.3.1 Cardinal Pivots

In this section, we discuss how Scarf's algorithm performs a cardinal pivot. Since $j_{t}$ is given, the basic feasible solution obtained after a cardinal pivot is performed is uniquely determined (see, e.g., [5, Section 3.2]), and we denote it by $x^{\prime}$. On the other hand, because of degeneracy, there may be multiple indices $j_{\ell}$ that can leave the basis, hence multiple basis corresponding to $x^{\prime}$. So our task is to find an appropriate $j_{\ell}$. In particular, we show the following.

Lemma 28. Consider a cardinal pivot of Scarf's algorithm, where $u_{1} \in \mathcal{L}$ and the separator is $m_{i}$. If $j_{t}$ is a man-disliked (w.r.t. D) valid edge, then we can always let the leaving column $j_{\ell}$ be either the loop $e_{i}$ or some woman-disliked edge (w.r.t. D).

Proof. We show that there is a basis $B^{\prime}$ corresponding to $x^{\prime}$ of the form $B \cup\left\{j_{t}\right\} \backslash\left\{j_{\ell}\right\}$, where $j_{\ell}$ is either the loop $\left(m_{i}, m_{i}\right)$ or a woman-disliked edge. Note that it may be that $x=x^{\prime}$ and / or that there are multiple basis corresponding to $x^{\prime}$.

By Lemma 21, $G_{B}$ is a forest with single loops. When a valid edge $e_{j_{t}}$ is added to our graph, one of the following happens:
(I) $e_{j_{t}}$ joins two different trees $T_{1}, T_{2}$ of $\left(V, E_{B}^{v}\right)$, as to form a larger tree $T$ with two loops.
(II) $e_{j_{t}}$ connects two nodes of a same tree $T_{1}$ of $\left(V, E_{B}^{v}\right)$.

Suppose (I) happens and consider the path $P$ connecting the two loops of $T$. Since $u_{1} \in \mathcal{L}$, we know that there is no valid edge incident to $m_{1}$ in $E_{D}$, thus $P$ is not incident to $m_{1}$. Therefore, all edges of $P$ are contained in $E_{D}$.
Claim 29. $P$ starts at $m_{i}$ with a loop and ends at a woman $\bar{w}$ with a loop. Moreover, suppose $P$ is incident to $p$ nodes, with

$$
\begin{equation*}
P=\left(m_{i_{1}}, m_{i_{1}}\right),\left(m_{i_{1}}, w_{i_{1}}\right),\left(w_{i_{1}}, m_{i_{2}}\right), \ldots,\left(m_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}\right),\left(w_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}\right) \tag{6}
\end{equation*}
$$

such that $m_{i_{1}}=m_{i}$ and $w_{i_{\frac{p}{2}}}=\bar{w}$. Then the edges of $P$ are disliked by

$$
m_{i_{1}}, m_{1}, w_{i_{1}}, m_{i_{2}}, w_{i_{2}}, \ldots, m_{i_{\frac{p}{2}-1}}, w_{i_{\frac{p}{2}-1}}, m_{i_{\frac{p}{2}}}, w_{i \frac{p}{2}}
$$

in this order.

Proof of Claim 29. Suppose $P$ is incident to $p$ nodes and has therefore $p-1$ valid edges, and $p+1$ edges in total (including loops). By Definition 18 and Corollary 25, each edge of $P$ is disliked by exactly one node from $V_{P} \cup\left\{m_{1}\right\}$. In particular, the $m_{1}$-disliked edge is on $P$, and by Proposition 23 it is exactly the rightmost column $\left(m_{i}, w\right) \in E_{D}$, where $m_{i}$ is the separator and $w$ is some woman. Hence, one of the loops on $P$ is $\left(m_{i}, m_{i}\right)$, and the first node of $P$ can without loss of generality be assumed to be $m_{i}$. On the other hand, again by Proposition [23, the last node of $P$ is a woman $\bar{w}$.

We have therefore obtained (6), where $m_{i_{1}}=m_{i}, w_{i_{1}}=w$ and $w_{i_{\frac{p}{2}}}=\bar{w}$. It is clear that $\left(m_{i_{1}}, m_{i_{1}}\right)$ is $m_{i_{1}}$-disliked, $\left(m_{i_{1}}, w_{i_{1}}\right)$ is $m_{1}$-disliked, and $\left(w_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}\right)$ is $w_{i_{\frac{p}{2}}}$-disliked. Then the edges of $P$ are disliked by

$$
m_{i_{1}}, m_{1}, w_{i_{1}}, m_{i_{2}}, w_{i_{2}}, \ldots, m_{i_{\frac{p}{2}-1}}, w_{i_{\frac{p}{2}-1}}, m_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}
$$

in this order.
Recall that we are letting some man-disliked valid edge $e_{j_{t}}$ enter the basis, with $e_{j_{t}} \in P$. Since $j_{t} \notin B$, we have $x_{j_{t}}=0$.

Case 1: $x_{j_{t}}^{\prime}=1$. We claim that there exists an $x$-augmenting path $P^{A}$ starting at $m_{i}$ with a loop, such that $e_{j_{t}} \in P^{A} \subset E_{D}$. In fact, consider the matchings $\mu_{x}$ and $\mu_{x^{\prime}}$. We have $e_{j_{t}} \notin \mu_{x}$ and $e_{j_{t}} \in \mu_{x^{\prime}}$. Now define the edge set $E_{\text {change }}=\left\{e \in E_{D} \mid x_{e} \neq x_{e}^{\prime}\right\}$. Then $e_{j_{t}} \in E_{\text {change }}$. Denote by $P^{A}$ the connected component in ( $V, E_{\text {change }}$ ) that contains $e_{j_{t}}$. Notice that every edge in $P^{A}$ still differs in $x$ and $x^{\prime}$, then for any $e \in P^{A}$, one of $x_{e}, x_{e}^{\prime}$ takes value 1 and the other takes value 0 .

Consider any node $v$ that belongs to $P^{A}$. There are exactly two edges in $P^{A}$ (including loops) incident to $v$, which are precisely the edges $v$ is incident to in $\mu_{x}$ and $\mu_{x^{\prime}}$ (notice that, by the definition of matching presented in Section 4, every node is incident to exactly one edge in $\mu_{x}$ and $\left.\mu_{x^{\prime}}\right)$. Since there is no cycle in $E_{D}$ and $P^{A}$ is connected, $P^{A}$ can only be a path. This path $P^{A}$ is $x$-alternating because any consecutive two edges with $x$-value 1,0 will cause the infeasibility of $x, x^{\prime}$, respectively. Furthermore, $P^{A}$ is $x$-augmenting since, by maximality, its starting and ending edges can only be loops. Hence $P^{A}$ is the desired $x$-augmenting path that contains $e_{j_{t}}$. By the definition of $x$-augmenting path, $P^{A}$ consists of at least two loops as the endpoints. By the structure of $E_{D}$, there is only one path in $E_{D}$ that contains more than one loop, which is $P$. Therefore, $P^{A}=P$, which implies that $P$ starts at $m_{i}$ with a loop because of Claim 29,

We are left to show that $B \cup\left\{j_{t}\right\} \backslash\{i\}$ is a feasible basis whose associated vertex is $x^{\prime}$. We first prove that $B^{\prime}=B \cup\left\{j_{t}\right\} \backslash\{i\}$ is a linearly independent set. We can observe this in the graph $G_{B^{\prime}}=\left(V, E_{B^{\prime}}\right)$. Recall that, by Lemma 21, $G_{B}$ has a forest with single loops structure. Adding $j_{t}$ and removing $i$ from $B$ keeps the structure, and each component of $G_{B^{\prime}}$ has exactly one loop. Hence, using again Lemma 21, $B \cup\left\{j_{t}\right\} \backslash\{i\}$ is a basis. In order to conclude that it corresponds to $x^{\prime}$, observe that the support of $x^{\prime}$ is contained in $D \cup\left\{1, j_{t}\right\} \backslash\{i\} \subset B \cup\left\{j_{t}\right\} \backslash\{i\}$.

Case 2: $x_{j_{t}}^{\prime}=0$. Hence, we are in a degenerate pivot and $x^{\prime}=x$. We claim that there is no $x$-augmenting path $P^{A}$ that contains $e_{j_{t}}$ in $E_{D}$. Otherwise, if such $P^{A}$ exists, define $y \in \mathbb{R}^{m}$ such that

$$
y_{e}=\left\{\begin{array}{cl}
1-x_{e}, & \text { if } e \in E_{P^{A}}, \\
x_{e}, & \text { if } e \notin E_{P^{A}} .
\end{array}\right.
$$

Since $P^{A}$ is $x$-augmenting, we have $A y=b$ and $y_{j_{t}}=1-x_{j_{t}}=1$. Let $v_{j}$ be one of the endpoints of $P^{A}$, then $y_{\left(v_{j}, v_{j}\right)}=1-x_{\left(v_{j}, v_{j}\right)}=0$. Define $B^{(y)}=B \cup\left\{j_{t}\right\} \backslash\{j\}$. Similarly to Case 1, we can deduce that $B^{(y)}$ is a basis of $A$, and by definition $B^{(y)} y_{B^{(y)}}=b$. Thus $y$ is a basic feasible solution, with $y_{j_{t}}=1$, a contradiction.

Notice that $E_{P} \subset E_{D}$, and $P$ is not $x$-augmenting. Then by Claim 29 there exists at least one edge $e \in P$, which is either $m_{i_{1}}$-disliked or woman-disliked such that $x_{e}^{\prime}=x_{e}=0$. Following
an argument similar to Case 1, we can let the edge $e_{j \ell}=e$ leave the basis as to obtain the basis $B^{\prime}=B \cup\left\{j_{t}\right\} \backslash\left\{j_{\ell}\right\}$ associated to $x^{\prime}=x$.

If (II) happens, then the entering valid edge $e_{j_{t}}$ creates a cycle $Q$ in $T$, where $Q$ must be even. $Q$ may be incident to at most one loop. By Lemma 21, the pivoting lets one of the valid edges in $Q$ exit the basis, in order to form $B^{\prime}$. Suppose $Q$ has $p$ (valid) edges incident to $p$ nodes, and recall that $p$ is even.

Claim 30. Every other edge of $Q$ is woman-disliked.
Proof of Claim 30. Case 1: $Q$ contains $m_{i}$. We claim that $\left(m_{i}, w\right)$ belongs to $Q$. If $\left(m_{i}, w\right)$ does not belong to $Q$, then by Corollary 25, the edges of $Q$ have to be disliked by nodes of $Q \backslash\left\{m_{i}\right\}$ (for $m_{i}$ dislikes $\left.\left(m_{i}, m_{i}\right)\right)$, contradicting Definition 18, Now let

$$
\begin{equation*}
Q=\left(m_{i_{1}}, w_{i_{1}}\right),\left(w_{i_{1}}, m_{i_{2}}\right), \ldots,\left(m_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}\right),\left(w_{i_{\frac{p}{2}}}, m_{i_{1}}\right) \tag{7}
\end{equation*}
$$

where $m_{i_{1}}=m_{i}$ and $w_{i_{1}}=w$. Since $\left(m_{i_{1}}, w_{i_{1}}\right)$ is $m_{1}$-disliked, then the edges on $Q$ are disliked by

$$
m_{1}, w_{i_{1}}, m_{i_{2}}, w_{i_{2}}, \ldots, m_{i_{\frac{p}{Z}-1}}, w_{i_{\frac{p}{Z}-1}}, m_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}
$$

in this order.
Case 2: $Q$ does not contain $m_{i}$. Let

$$
Q=\left(m_{i_{1}}, w_{i_{1}}\right),\left(w_{i_{1}}, m_{i_{2}}\right), \ldots,\left(m_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}\right),\left(w_{i_{\frac{p}{2}}}, m_{i_{1}}\right) .
$$

Since the unique $m_{1}$-disliked edge does not belong to $Q$ (otherwise $m_{i}$ is incident to $Q$ and we are in case 1 ), the edges on $Q$ must be disliked by man, woman alternatively.

By a similar argument as in (I), we can obtain the desired result. In detail, if all woman-disliked edges on $Q$ have $x$-value 1 , then $Q$ is $x$-alternating, with all woman-disliked edges on $Q$ having $x$-value 1 , and all man-disliked edges having $x$-value 0 . Define $y \in \mathbb{R}^{m}$ as

$$
y_{e}=\left\{\begin{array}{cl}
1-x_{e}, & \text { if } e \in E_{Q}, \\
x_{e}, & \text { if } e \notin E_{Q} .
\end{array}\right.
$$

It is not difficult to check that $y$ is a basic feasible solution with $y_{j_{t}}=1$, and $B^{\prime}=B \cup\left\{j_{t}\right\} \backslash\left\{j_{\ell}\right\}$ is a basis corresponding to $y$, where $j_{\ell}$ is chosen to be any woman-disliked edge of $Q$. If conversely there is a woman-disliked edge $e$ with $x_{e}=0$, then we have $x=x^{\prime}$ and let $e_{j_{\ell}}=e, B^{\prime}=B \cup\left\{j_{t}\right\} \backslash\left\{j_{\ell}\right\}$, and the claim follows analogously.

Remark 31. In the proof of Lemma 28, we construct a vector $y$ several times, as the symmetric difference of the matching $\mu_{x}$ and the alternating subgraph ( $P$ or $Q$ ). Formally, $y$ is the characteristic vector of $\mu_{x} \Delta E_{P}$ (resp., $\mu_{x} \Delta E_{Q}$ ). In the theory of bipartite matching, it is well-known that a matching $\mu$ is maximal if and only if no $\mu$-augmenting path exists. We can see the similarity in our redefined matching problem.

### 5.3.2 Ordinal Pivots

This is the continuation of the previous section. We discuss how Scarf's algorithm finds an entering column $j^{*}$ when $j_{\ell}$ leaves $D$, as to form $D^{\prime}$. We still assume $u_{1} \in \mathcal{L}$ and that $m_{i}$ is the separator in D.

If $e_{j_{\ell}}$ is $v_{i_{\ell}}$-disliked in $D$, then according to Definition 11 column $j_{r}$ satisfies that $c_{i_{\ell}, j_{r}}$ is the second least element on row $i_{\ell}$ in $C_{D}$. Recall that we call $j_{r}$ (resp., $e_{j_{r}}$ ) a reference column (resp., edge). Suppose the reference edge $e_{j_{r}}$ is $v_{i_{r}}$-disliked w.r.t. $D$. Let $j^{*}$ be the entering column in the ordinal pivot. The following observation follows easily from the last definition.

Lemma 32. In the ordinal basis $D^{\prime}=D \backslash\left\{j_{\ell}\right\} \cup\left\{j^{*}\right\}$, we have that $e_{j_{r}}$ is $v_{i_{\ell}}$-disliked and $e_{j^{*}}$ is $v_{i_{r}}$-disliked.

Using the notations above, the following property plays a key role in the ordinal pivot:
Lemma 33. $c_{i_{\ell}, j_{r}} \in \mathcal{M}$.
Proof. We can claim this by showing $c_{i_{\ell}, j_{r}} \notin \mathcal{S} \cup \mathcal{L} \cup \mathcal{X} \mathcal{L}$.
If $c_{i_{\ell}, j_{r}}=0$, then $c_{i_{\ell}, j_{r}}<c_{i_{\ell}, j_{r}}=0$, a contradiction. Thus $c_{i_{\ell}, j_{r}} \notin \mathcal{S}$.
If $c_{i_{\ell}, j_{r}} \in \mathcal{L} \cup \mathcal{X} \mathcal{L}$, then $c_{i_{\ell}, j_{r}} \geq k+1$. Since $c_{i_{r}, j_{\ell}}$ is the second least element from $\left\{c_{i_{r}, j}: j \in D\right\}$, there is at most one edge in $E_{D}$ incident to $v_{i_{r}}$, which, if exists, must be the leaving edge $e_{j_{\ell}}$. From the discussion in Section 5.3, $B^{\prime}=D \cup\{1\} \backslash\{j \ell\}$ is a basis visited by Scarf algorithm. As a result, however, there is no edge incident to $v_{i_{r}}$ in $E_{B^{\prime}}$, which contradicts the feasibility of $B^{\prime}$. Thus $c_{i_{\ell}, j_{r}} \in \mathcal{M}$.

From Remark 32 and Lemma 33 we can immediately deduce that $e_{j_{r}}$ is not a loop.
Corollary 34. In any ordinal pivot, the reference edge $e_{j_{r}}$ is a valid edge.
The ordinal pivot will change the utility vector $u$, but only in components $u_{i_{\ell}}$ and $u_{i_{r}}$. The following lemma translates Lemma 12 i.e., some basic facts on the mechanics of Scarf's algorithm, into graphic language.

Lemma 35. The ordinal pivot gives $(D, u) \rightarrow\left(D^{\prime}, u^{\prime}\right)$, where $u_{i_{\ell}}^{\prime}>u_{i_{\ell}}, u_{i_{r}}^{\prime}<u_{i_{r}}$, and $u_{i}^{\prime}=$ $u_{i}$, for $i \neq i_{\ell}, i_{r}$. Correspondingly, we have that:

1. $v_{i_{\ell}}$ dislikes the leaving edge $e_{j_{\ell}}$ in $D$ and dislikes the reference edge $e_{j_{r}}$ in $D^{\prime}$.
2. $v_{i_{r}}$ dislikes the reference edge $e_{j_{r}}$ in $D$ and dislikes the entering edge $e_{j^{*}}$ in $D^{\prime}$.
3. Any other node dislikes the same edge in $D$ and $D^{\prime}$.

As a continuation of Lemma 28, we will see that a leaving loop $e_{i}$ or a leaving woman-disliked edge will always return a new man-disliked valid edge $e_{j^{*}}$ in an ordinal pivot.

Lemma 36. Assume $u_{1} \in \mathcal{L},\{2, \ldots, k\} \cap D \neq \emptyset$, and that the leaving edge $e_{j_{\ell}}$ is woman-disliked (w.r.t. D). Then:

1. $v_{i_{\ell}}$ is a woman and $v_{i_{r}}$ is a man;
2. $e_{j^{*}}$ is man-disliked in $D^{\prime}$;
3. the woman $v_{i_{\ell}}$ 's utility increases, i.e., $u_{i_{\ell}}^{\prime}>u_{i_{\ell}}$, and any other woman's utility does not change.

Proof. Since $e_{j_{\ell}}$ is woman-disliked and $v_{i_{\ell}}$-disliked (w.r.t. $D$ ), $v_{i_{\ell}}$ can only be a woman. Since by Corollary 34, the reference edge $e_{j_{r}}$ is a valid edge and is not disliked by its incident woman $v_{i_{\ell}}$, then it is disliked by a man, by Corollary 25. Therefore, $v_{i_{r}}$ is a man. This shows 1.

The ordinal pivot introduces a new $v_{i_{r}}$-disliked edge $e_{j^{*}}$ into $D^{\prime}$. Recall that $m_{i}$ is the separator in $D$. Assume by contradiction that $e_{j^{*}}$ is a loop. Then $e_{j^{*}}=\left(v_{i_{r}}, v_{i_{r}}\right)$. By Lemma 27, the new ordinal basis $D^{\prime}$ must have a separator. We claim that this separator can only be $m_{i-1}$. Indeed, loops corresponding to $e_{j}$ for $j \geq i$ belong to $D^{\prime}$. Hence, one of $m_{1}, \ldots, m_{i-1}$ is the separator in $D^{\prime}$. However, if $m_{j}$ is the separator for some $j \leq i-2$, then we contradict Lemma 27(i). Hence, $m_{i-1}$ is the separator.

By definition, all valid edges incident in $D$ to $m_{i}$ must leave $D$. By Lemma 27(ii), one such edge is $m_{1}$-disliked. This contradicts the fact that $e_{j_{\ell}}$ is woman-disliked. Therefore, $e_{j^{*}}$ is a valid edge. This shows 2 .

As for the change of utility vector, we have by Lemma 35, part 1 ,

$$
u_{i_{\ell}}^{\prime}=c_{i_{\ell}, j_{r}}>c_{i_{\ell}, j_{r}}=u_{i_{\ell}} .
$$

Since $v_{i_{r}}$ is a man, it follows from Lemma [35, part 3 that every other woman's utility does not change. This shows 3 .

We remark that, for the following fundamental lemma to holds, we need the extra properties of matrix $C$ that distinguish it from the generic matrix defined by Biró and Fleiner [7], as discussed in Section 5.1.

Lemma 37. Assume $u_{1} \in \mathcal{L}$ and that the leaving edge $e_{j_{\ell}}$ is the loop $\left(m_{i}, m_{i}\right)$. Then:

1. $v_{i_{\ell}}=m_{i}, v_{i_{r}}=m_{1}$;
2. Let $u^{\prime}$ be the utility vector of the ordinal basis $D^{\prime}$ obtained wal pivot. Then a new $m_{1}$-disliked (w.r.t. $D^{\prime}$ ) valid edge $e_{j^{*}}$ enters and:
(i) If $2 \leq i<k, e_{j^{*}}$ is incident to $m_{i+1}$, and

$$
\begin{gathered}
u_{1}^{\prime}<u_{1}, \text { but still } u_{1}^{\prime} \in \mathcal{L} . \\
u_{i}^{\prime}=u_{i}, \text { for } k<i \leq 2 k .
\end{gathered}
$$

(ii) If $i=k, e_{j^{*}}$ is incident to $m_{1}$, and

$$
\begin{gathered}
u_{1}^{\prime}<u_{1}, \text { and } u_{1}^{\prime} \in \mathcal{M} . \\
u_{i}^{\prime}=u_{i}, \text { for } k<i \leq 2 k .
\end{gathered}
$$

Proof. Loop $\left(m_{i}, m_{i}\right)$ is $m_{i}$-disliked w.r.t. $D$, thus $v_{i_{\ell}}=m_{i}$. By Lemma 33, $c_{i_{\ell}, j_{r}} \in \mathcal{M}$, thus the edge $e_{j_{r}}$ is incident to $m_{i}$. By Lemma 27(ii), the rightmost column in $D$ is also incident to $m_{i}$. Moreover, the rightmost column in $D$ is exactly $j_{r}$ because $m_{i}$ 's second worst choice $e_{j_{r}}$ between all edges is the worst choice between all valid edges, which corresponds to the rightmost column. Since $u_{1} \in \mathcal{L}$, the rightmost column $j_{r}$ is $m_{1}$-disliked, hence $v_{i_{r}}=m_{1}$. This shows 1 .

Notice that by Lemma 26, there is a woman $\bar{w}$ whose loop $(\bar{w}, \bar{w}) \in E_{D}$. We use this fact to prove part 2..
(i) If $2 \leq i<k$, then by Definition [11, the entering column $j^{*}$ satisfies

$$
\begin{equation*}
c_{h j^{*}}>\bar{u}_{\ell} \text { for all } h \neq 1 \tag{8}
\end{equation*}
$$

Consider the edge $e_{j}=\left(m_{i+1}, \bar{w}\right)$. Column $c_{j}$ satisfies the above condition and $j \notin D$ (By Lemma 27(ii), the rightmost column in $D$ is incident to $m_{i}$, thus no valid edge incident to $m_{i+1}$ belongs to D). Therefore,

$$
\begin{equation*}
u_{1}^{\prime}=c_{1 j^{*}} \geq c_{1,\left(m_{i+1}, \bar{w}\right)} \tag{9}
\end{equation*}
$$

Thus $u_{1}^{\prime} \in \mathcal{L}$. Notice that $u_{1}^{\prime}<u_{1}$ must hold, otherwise $c_{j^{*}}>u$, a contradiction. Suppose $e_{j_{r}}=\left(m_{i}, w\right)$, then

$$
\begin{equation*}
u_{1}^{\prime}<u_{1}=c_{1,\left(m_{i}, w\right)} . \tag{10}
\end{equation*}
$$

(9) and (10) imply that edge $e_{j^{*}}$ is incident to either $m_{i}$ or $m_{i+1}$. Suppose the former happens, then by (10), $c_{1, j^{*}}<c_{1,\left(m_{i}, w\right)}=c_{1, j_{r}}$, thus column $j^{*}$ is on the right of column $j_{r}$ in $C$, which implies

$$
c_{i, j^{*}}<c_{i, j_{r}}=\bar{u}_{i} .
$$

This contradicts (8) for $\ell=i$. Therefore, $e_{j^{*}}$ must be incident to $m_{i+1}$.
This ordinal pivot only changes the row minimizer of $m_{i}$ and $m_{1}$. Since no row minimizer corresponding to a women changes, we have

$$
u_{i}^{\prime}=u_{i}, \text { for } k<i \leq 2 k .
$$

(ii) If $i=k$, then consider the edge $e_{j}=\left(m_{1}, \bar{w}\right)$. With a similar argument as in (i) we argue that $e_{j^{*}}$ can only be incident to $m_{1}$. Now suppose $e_{j^{*}}=\left(m_{1}, w^{\prime}\right)$. In this stage, a valid edge incident to $m_{1}$ first enters ordinal basis. Notice that

$$
u_{1}^{\prime}=c_{1, j^{*}}=c_{1,\left(m_{1}, w^{\prime}\right)} \in \mathcal{M} .
$$

Also, the utility of each woman does not change, which completes the proof.

### 5.4 Convergence

Recall that an ordinal pivoting is uniquely defined, once that a column entering the current ordinal basis has been selected. The cardinal pivoting rule described in Algorithm 1 will, in a polynomial number of iterations, lead to the convergence of Scarf's algorithm.

Recall that, at the first iteration of Scarf's algorithm, we have $u_{i} \in \mathcal{L}$.

```
Algorithm 1 Cardinal pivoting rule
    Let \(B\) be the current feasible basis, \(D\) be the current ordinal basis with utility vector \(u\), and \(j_{t}\)
    the man-disliked (w.r.t. \(D\) ) edge that is going to enter \(B\).
    if \(u_{1} \in \mathcal{L}\) then
        Let \(m_{i}\) be the separator.
    else
        Set \(i=1\).
    end if
    if \(e_{i}=\left(v_{i}, v_{i}\right)\) is a candidate to leave the basis then
        Let \(e_{i}\) leave the basis.
    else
Choose any woman-disliked valid edges (w.r.t. \(D\) ) that is a candidate to leave the basis as the variable that leaves the basis.
    end if
```

To prove polynomial-time convergence, we start with an auxiliary lemma.

Lemma 38. If Scarf's algorithm iteratively applies Algorithm 1 to perform a cardinal pivot while $u_{1} \in \mathcal{L}$, then we obtain an ordinal basis with $u_{1} \in \mathcal{M}$ after $O\left(k^{2}\right)$ steps. More in detail, at every step, we weakly increase the subscript of the separator (where we identify $k+1$ with 1 ) and the total utility of women $\quad \sum_{w \in W} u_{w}$, and strictly increase at least one of them.
Proof. Suppose $k \geq 2$, else the statement is trivial. By construction, $B_{0}=\{1,2, \ldots, n\}$ and $D_{0}=\{3 k+1,2,3, \ldots, n\}$, since $c_{1,3 k+1}$ is the maximum entry in the first row outside the first $n$ columns, (see Example 19 for an illustration).

We first show by induction on the number of iterations that while $u_{1} \in \mathcal{L}$, the input to Algorithm 1 is well-defined, and moreover, there is always a variable candidate to leave the basis that is either of the form $e_{i}$ (where $m_{i}$ is the current separator) or a woman-disliked edge (w.r.t. the current ordinal basis $D$ ). Recall that an iteration is defined as the change of both an ordinal and a feasible basis.

For the basic step, note that $u_{1}=c_{1,3 k+1} \in \mathcal{L}, 1 \notin D$ and $2 \in D$. By Lemma 27(ii) $m_{2}$ is the separator. At the first iteration, we execute a cardinal pivot to let $e_{3 k+1}$ enter $B_{0}$, which as argued above is man-disliked. Hence, the input to Algorithm $\square$ is well-defined. The second part of the statement follows from Lemma 28,

For the inductive step, let us investigate the generic iteration $g$-th iteration, with $g \in \mathbb{N}$, of the algorithm. By inductive hypothesis, the input to Algorithm $\prod$ in iteration $g-1$ is well-defined, and the leaving variable is chosen to be either $e_{i}$ (where $m_{i}$ is the current separator) or a woman-disliked edge (w.r.t. $D$ ). We can then apply Lemma 36 or Lemma 37 to conclude that the edge entering the current feasible basis $B$ is man-disliked. Since every almost-feasible basis has a separator (see Proposition [23), the input to Algorithm 1 is well-defined. The second part of the statement follows from Lemma 28.

Hence, Scarf's algorithm that iteratively applies Algorithm $\square$ for choosing a cardinal pivoting rule is well-defined. Let us now argue about its convergence to an ordinal basis with $u_{1} \in \mathcal{M}$. Note that during an ordinal pivot, either we move the separator from $i$ to $i+1$ (when $i=n$, define $i+1=1$ ), while all women's utility stay constant (Lemma 37, part 2), or the separator does not move, but a woman's utility increases, while all other stay constant (Lemma 36, part 3). Throughout the algorithm, the utility of a woman is contained in $\mathcal{S} \cup \mathcal{M}$. Hence, after $O\left(k^{2}\right)$ iterations, we must have that the separator becomes $m_{1}$, which implies $u_{1} \in \mathcal{M}$ (see Lemma 37, part 2).

Consider the first iteration when $u_{1} \notin \mathcal{L}$. Then $u_{1} \in \mathcal{M}$ and, before that, no valid edge incident to $m_{1}$ occurs in the intermediate feasible basis $B$ (since $u_{1} \in \mathcal{L}$ throuand wghout the first part of the algorithm), hence $m_{1}$ has never been matched in any matching corresponding to the feasible basis visited by Scarf's algorithm. By Lemma 38, in the next iteration we start with $u_{1} \in \mathcal{M}$.

Now assume $u_{1} \in \mathcal{M}$, and define $m_{1}$ to be the separator. We continue with a cardinal pivot to let some $m_{1}$-disliked edge enter our basis. Using Algorithm [1, we can repeat arguments similar to Lemma 27 Lemma 28, Lemma 36 and Lemma 37 to conclude the following. Details are given in Appendix A.2
Lemma 39. Consider an iteration of Scarf's algorithm that followed Algorithm $\mathbb{\square}$ until $u_{1} \in \mathcal{M}$. Let $D$ be the current almost-feasible ordinal basis assume we have $u_{1} \in \mathcal{M}$. Then
(i) $1,2, \ldots, k \notin D$.
(ii) The unique $m_{1}$-disliked edge in $D$ corresponds to $\left(m_{1}, w\right)$ for some $w \in W$.
(iii) In any cardinal pivot, suppose $B$ is our current feasible basis and $D=B \cup\left\{j_{t}\right\} \backslash\{1\}$ is the associated ordinal basis. If $e_{j_{t}}$ is a man-disliked valid edge, then we can find either the loop $e_{1}$ or some woman-disliked edge $e_{j_{\ell}}$ to leave $B$, and we obtain $B^{\prime}=B \cup\left\{j_{t}\right\} \backslash\left\{j_{\ell}\right\}$ as a new feasible basis.
(iv) If the leaving edge $e_{j_{\ell}}$ is woman-disliked, then we do not terminate. In the following ordinal pivot, let $e_{j_{r}}$ be the reference edge. Then $v_{i_{\ell}}$ is a woman and $v_{i_{r}}$ is a man. A new man-disliked valid edge $e_{j^{*}}$ enters $D$. Moreover, denote $u^{\prime}$ as the utility vector of the new ordinal basis $D^{\prime}$, then

$$
u_{i_{\ell}}^{\prime}>u_{i_{\ell}}, \quad u_{i_{r}}^{\prime}<u_{i_{r}}, \quad u_{i}^{\prime}=u_{i} \text { for } i \neq i_{\ell}, i_{r} .
$$

In particular, $\sum_{w \in W} u_{w}$ strictly increases.
(v) If the leaving edge is the loop $e_{1}$, then column 1 leaves $B$, then we obtain $B^{\prime}=D$, which terminates the algorithm.

These principles will indicate us to continue our anti-cycling pivots and end with (v) after $O\left(k^{2}\right)$ steps. Combining Lemma 38 with Lemma 39 and bounding, in the case $u_{1} \in \mathcal{M}$, the number of steps in a similar fashion as it was done when $u_{1} \in \mathcal{L}$, we can bound the total running time of the algorithm.
Theorem 40. If Scarf's algorithm iteratively applies Algorithm $\mathbb{\square}$ to perform a cardinal pivot, then it converges after $O\left(n^{2}\right)$ steps.

## 6 Convergence Through a Perturbation of the Bipartite Matching Polytope

It is common in literature to run Scarf's algorithm preceded by a perturbation, so that the resulting polytope is non-degenerate [6, 7]. Recall that, in this case, the behavior of Scarf's algorithm is univocally defined, see Section 1.1. In this section, we give another perspective on our pivot rule connecting it to the perturbation approach.

Notice that the bipartite matching polytope given in form (11) is highly degenerate. For $A, b$ as given in Section 5.1, we define a non-degenerate polytope

$$
\begin{equation*}
\left\{x \in \mathbb{R}_{\geq 0}^{m}: A x=b+b(\epsilon)\right\}, \tag{11}
\end{equation*}
$$

where $b(\epsilon) \in \mathbb{Q}_{\geq 0}^{n}$ is a parameterized vector defined by

$$
b(\epsilon):=(\underbrace{\epsilon^{k+1}, \epsilon^{k+2}, \ldots, \epsilon^{2 k}}_{\text {men }}, \underbrace{\epsilon, \epsilon^{2}, \ldots, \epsilon^{k}}_{\text {women }})^{T} .
$$

Observation 1. In (11), we have that $\sum_{m \in M} b_{m}(\epsilon)<b_{w}(\epsilon)$ for every woman $w$. In particular, the right-hand side of each constraint corresponding to a woman is strictly larger than the right-hand side of each constraint corresponding to a man.

The following is a classical fact of linear algebra, see, e.g., [5, Exercise 3.15].
Lemma 41. There exists some $\epsilon^{*}>0$, such that for any fixed $\epsilon$ with $0<\epsilon<\epsilon^{*}$
(i) All basic feasible solutions to the polytope defined in (11) are nondegenerate. That is, every basic feasible solution $x$ of (11) has exactly $n$ strictly positive entries.
(ii) Every basis that is feasible for (11) is feasible for the original polytope.

For any fixed $b(\epsilon)$ that makes the polytope nondegenerate, the iterations of Scarf's algorithm generate a sequence of Scarf pairs

$$
\begin{equation*}
\left(B_{0}, D_{0}\right) \rightarrow\left(B_{1}, D_{1}\right) \rightarrow \cdots \rightarrow\left(B_{I}, D_{I}\right) \tag{12}
\end{equation*}
$$

Recall that such that each pair contains two $n$-sets with $\left|B_{I} \cap D_{I}\right|=n-1$ for $I<N$ and $B_{N}=D_{N}$.
Our pivot rule (Algorithm (1) can be captured by a specific perturbation:

Theorem 42. There exists $\epsilon>0$ such that an execution of Scarf's algorithm on the original bipartite matching polytope with pivot rule given by Algorithm 1 gives the same sequence (12) as running it on its perturbation (11).

Proof. Fix $\epsilon=\min \left\{\frac{1}{2 n+1}, \epsilon^{*}\right\}$, where $\epsilon^{*}$ makes Lemma 41 valid. Then (11) is nondegenerate.
Let $B$ be a feasible basis of (11), corresponding to vertex $x^{\epsilon}$. By Lemma 41 $B$ is also feasible for the original bipartite matching polytope, and let $x$ be the basic feasible solution corresponding to it. If $j \in B$,

$$
\left|\left(x^{\epsilon}-x\right)_{j}\right|=\left|\left(A_{B}^{-1} b(\epsilon)\right)_{j}\right|=\left|\sum_{i=1}^{n}\left(A_{B}^{-1}\right)_{j i} b_{i}(\epsilon)\right| \leq \sum_{i=1}^{n}\left|\left(A_{B}^{-1}\right)_{j i}\right|\left|b_{i}(\epsilon)\right| \leq \frac{1}{2 n+1} \sum_{i=1}^{n}\left|\left(A_{B}^{-1}\right)_{j i}\right|<\frac{1}{2},
$$

where the last inequality uses the fact that $A$ is totally unimodular.
Combining the inequality above and the fact that $x_{j}=x_{j}^{\epsilon}=0$ for $j \notin B$, we have
Claim 43. For any $j \in[n], x_{j}=1$ if and only if $x_{j}^{\epsilon}>\frac{1}{2}$.
Notice that Lemma 21 is independent of the right-hand side, thus the graph representation $G_{B}$ of polytope (11) still has the forest with single loops structure. Now consider a tree $T$ with single loop from the forest. We define the root of $T$ as the node $r$ such that its loop belongs to $E_{B}$. Then $r$ is uniquely defined.

We now investigate properties of the vector $x$ restricted to $T$. For any node $v$, recall that $\operatorname{deg}_{T}(v)$ is defined as the number of edges (including loops) incident to $v$ in $T$. We call a node $v$ of $T$ a leaf if $v \neq r$ and $\operatorname{deg}_{T}(v)=1$. $x_{e}$ for each edge $e$ incident to a leave of $T$ is uniquely defined, and we can then inductively define the other values of $x_{e}$ for each edge $e$ of $T$.

Our perturbation imply the following property of leaves.
Claim 44. If $v$ is a leaf of $T$, then $v$ is a man.
Proof of Claim 44. If a woman $w$ has degree 1 in $T$ and $(w, w) \notin E_{B}$, then consider the only edge
 contradicts the feasibility of $B$ since we require that any edge in $E_{B}$ incident to $m$ has nonnegative $x^{\epsilon}$-value.

Claim 45. For any woman $w$, we have $\operatorname{deg}_{T}(w)=1$ or $\operatorname{deg}_{T}(w)=2$.
Proof of Claim 45. If $T$ only contains $w$, then we have $\operatorname{deg}_{T}(w)=1$ since the loop $(w, w)$ is the only edge on $T$. For the rest of the proof, we assume $T$ has at least two nodes.

So assume that $T$ contains at least two nodes. If $w$ is not a root, then from Claim 44, we know $\operatorname{deg}_{T}(w) \neq 1$. Suppose $\operatorname{deg}_{T}(w) \geq 3$. Since (11) is integral when $b(\epsilon)$ is the 0 vector, there exists at least two distinct men $m_{a}, m_{b}$ such that $e_{j}=\left(m_{a}, w\right) \in E_{T}, e_{k}=\left(m_{b}, w\right) \in E_{T}$, and $x_{j}=0, x_{k}=0$. For $m_{a}$, there exists another edge $e_{j^{\prime}}$ incident to $m_{a}$ such that $x_{j^{\prime}}=1$ (again, using the integrality of the polytope). The other endpoint of $e_{j^{\prime}}=\left(m_{a}, w_{a}\right)$ is a woman $w_{a}$ but cannot be a leaf by Claim44, which implies there exists at least one man $m_{a^{\prime}}$ such that $e_{j^{\prime \prime}}=\left(m_{a^{\prime}}, w_{a}\right) \in E_{T}$ with $x_{j^{\prime \prime}}=0$ (again using integrality). Continue extending this path, we have a sequence ( $w, m_{a}, w_{a}, m_{a^{\prime}}, \ldots$ ) with every edge consisting of two consecutive nodes presenting in $T$. This sequence will not end unless it contains the root. Notice that the same argument holds for the sequence ( $w, m_{b}, w_{b}, m_{b^{\prime}}, \ldots$ ) defined in the same way. Sincewhich is a one root in $T$ and $T$ has no cycles, we must have $m_{a}=m_{b}$, a contradiction.

If $w$ is the root, then $\operatorname{deg}_{T}(w) \neq 1$ since $T$ is not a singleton. Suppose $\operatorname{deg}_{T}(w) \geq 3$. Then there exists a man $m_{a}$ such that $e_{j}=\left(m_{a}, w\right) \in E_{T}$ and $x_{j}=0$ (again by integrality). Similarly as
before, we can continue this path with another edge $e_{j^{\prime}}=\left(m_{a}, w_{a}\right)$ such that $x_{j^{\prime}}=1$, and so on. We obtain a sequence $\left(w, m_{a}, w_{a}, \ldots\right)$. Since $T$ is finite, we must end the sequence with a root, but since there is only one root $w$ on $T$ and $T$ has no cycles, we obtain a contradiction.

Using Claim 45 and Claim 44, we conclude that the vector $x$ restricted to $T$ has a very regular structure, discussed in the next claim.
Claim 46. For any tree $T$ from the forest with single loops structure of $G_{B}$, we have that:

1. Any path on $T$ starting from the root $r$ is $x$-alternating.
2. If the root is a man $m$, then the loop has $x_{(m, m)}=1$.
3. If the root is a woman $w$, then the loop has $x_{(w, w)}=1$ if $T$ only has one node $w$. Else if $T$ has at least two nodes, then $x_{(w, w)}=0$.

## Proof of Claim 46.

1. If $T$ only contains one node, then by the definition in Section 4 it is clearly $x$-alternating.

Suppose $T$ has at least two nodes. Fix a path $P$ in $T$ such that $r \in V_{P}$. Notice that by Claim 45, a woman cannot be incident to two edges $e, e^{\prime}$ with $x_{e}=x_{e^{\prime}}=0$ because of feasibility. Hence, if along the path $P$ we have two consecutive 0 's, i.e., $e, e^{\prime} \in E_{P}, e \cap e^{\prime} \neq$ and $x_{e}=x_{e^{\prime}}=0$ then $e \cap e^{\prime}=m$ for some man. By feasibility, we can find a third edge $\tilde{e}$ such that $m \in \tilde{e}$ and $x_{\tilde{e}}=1$. However, we can start from $\tilde{e}$ and find another $x$-alternating path $\tilde{P}$ starting at $m$ and edge-disjoint from $P$ because, by feasibility, we can always find a successor of a man with an incoming 0 -edge, and, by degree count, find a successor of a woman. Similar to the proof of Claim 45, the path $\tilde{P}$ must end at the root, which is a contradiction since $E_{P} \cap E_{\tilde{P}}=\emptyset$. This shows $\square$.

2 and 3. Consider the following parity argument: When $T$ is not a singleton, then there must exist a leaf, we $m$ by Claim 45, $m$ creates an $x$-alternating path starting from $e$ with $m \in e$ and $x_{e}=1$, which ends at the root. Then 2,3 hold.

By Claim 46, $T$ must be one of the following four types, presented in Figure 6
(a) A tree rooted at a man $m$, with at least two nodes. $x_{(m, m)}=1$ and every path from the root to a leaf is $x$-alternating.
(b) A tree rooted at a woman $w$, with at least two nodes. $x_{(w, w)}=0$ and every path from the root to a leaf is $x$-alternating.
(c) A singleton man $m$ with $x_{(m, m)}=1$.
(d) A singleton woman $w$ with $x_{(w, w)}=1$.

We now investigate $x^{\epsilon}$. Our key observation is that, for the $T$ of type (a) and type (b), the $x^{\epsilon}$-value has a monotonic property:

Claim 47. Let $T$ be a tree of $G_{B}$ of type (a) or type (b). Fix any path $P=r, f_{0}, r, f_{1}, v_{1}, f_{2}, \ldots, f_{p}, v_{p}$ from the root $r$ to a leaf, where $f_{0}=(r, r)$ is the loop, $f_{i}=\left(v_{i-1}, v_{i}\right)$ for $i \in[p]$, and $v_{p}$ is a man by Claim 44 Then we have for $0 \leq i<j \leq p$ :

1. If $x_{f_{i}}=x_{f_{j}}=0$, then $\frac{1}{2} \geq x_{f_{i}}^{\epsilon}>x_{f_{j}}^{\epsilon}>0$.
2. If $x_{f_{i}}=x_{f_{j}}=1$, then $\frac{1}{2}<x_{f_{i}}^{\epsilon}<x_{f_{j}}^{\epsilon}$. Moreover, $x_{f_{j}}^{\epsilon}>1$ if and only if $j=p$.


Figure 6: An illustration of four types of $T$. They are corresponding to type (a),(b),(c)(d) from left to right. The solid edges (resp. dotted edges) are associated to $x$-value 1 (resp. 0). Notice that every path from the root to the leaf is $x$-alternating.

Proof of Claim 47. Let us consider a simple case when $T$ itself is a path. Hence, $P=T$. Then we can directly obtain the value of $x^{\epsilon}$ by reversing the order in which nodes are visited in $P$. Indeed, by feasibility we have:

$$
\begin{aligned}
& x_{f_{p}}^{\epsilon}=1+b_{v_{p}}(\epsilon), \\
& x_{f_{f-1}}^{\epsilon}=1+b_{v_{p-1}}(\epsilon)-x_{f_{p}}^{\epsilon}=b_{v_{p-1}}(\epsilon)-b_{v_{p}}(\epsilon), \\
& x_{f_{p-2}}^{\epsilon}=1+b_{v_{p-2}}(\epsilon)-x_{f_{p-1}}^{\epsilon}=1+b_{v_{p-2}}(\epsilon)+b_{v_{p}}(\epsilon)-b_{v_{p-1}}(\epsilon),
\end{aligned}
$$

We can use induction to obtain the formula

$$
x_{f_{p-k}}^{\epsilon}=\left\{\begin{array}{ll}
1-\left(\sum_{\ell=1}^{k / 2} b_{v_{p-k+2 \ell-1}}(\epsilon)-\sum_{\ell=0}^{k / 2} b_{v_{p-k+2 \ell}}(\epsilon)\right), & \text { if } k \text { is even }  \tag{13}\\
\sum_{\ell=1}^{(k+1) / 2} b_{v_{p-k+2 \ell-1}}(\epsilon)-\sum_{\ell=0}^{(k-1) / 2} b_{v_{p-k+2 \ell}}(\epsilon), & \text { if } k \text { is odd }
\end{array} .\right.
$$

Notice that $k$ is even iff $x_{f_{p-k}}=1$, also iff $v_{p-k}$ is a man. Formula (13) can be translated into a more intuitive form. For a generic tree $T$ (not necessarily a path), we say $v$ (resp. e) is a node (resp. edge) after $v_{k}$ on $T$ if there is a path $P=r, f_{0}, r, f_{1}, v_{1}, f_{2}, \ldots, f_{p}, v_{p}$ so that, when we remove edge $f_{k}$, then $v$ (resp. e) and $v_{k}$ are are still connected. Notice that $v_{k}$ is always after $v_{k}$ itself.

Going back to the case when $T$ is a path, the following can be easily deduced from (13). If $x_{f_{k}}=1$, then

$$
\begin{equation*}
x_{f_{k}}^{\epsilon}=1+\sum_{m \text { is a man after } v_{k}} b_{m}(\epsilon)-\sum_{w \text { is a woman after } v_{k}} b_{w}(\epsilon), \tag{14}
\end{equation*}
$$

else if $x_{f_{k}}=0$, then

$$
\begin{equation*}
x_{f_{k}}^{\epsilon}=\sum_{w \text { is a woman after } v_{k}} b_{w}(\epsilon)-\sum_{m \text { is a man after } v_{k}} b_{m}(\epsilon) . \tag{15}
\end{equation*}
$$

It is not hard to see (14) and (15) hold in general. In fact, we say that $T$ has a branching (appearing at $v$ ) if there is $v$ such that $\operatorname{deg}_{T}(v) \geq 3$. We show that the two formulas are still true. We first observe that by Claim 45, the branching can only appear at some man. Let $v_{k}$ be a man with $\operatorname{deg}_{T}\left(v_{k}\right) \geq 3$. Assume every edge after $v_{k}$ on $T$ makes (14) and (15) true. Denote $P^{(\alpha)}=$
$r, f_{0}, r, f_{1}, v_{1}, f_{2}, \ldots, f_{k}, v_{k}, f_{k+1}^{(\alpha)}, v_{k+1}^{(\alpha)}, \ldots$ as the paths that pass through $v_{k}$ with distinct branches $f_{k+1}^{(\alpha)}$ for $\alpha=1, \ldots, \operatorname{deg}_{T}\left(v_{k}\right)-1$. Then by feasibility,

$$
\begin{align*}
& x_{f_{k}}^{\epsilon}=1+b_{v_{k}}(\epsilon)-\sum_{\alpha=1}^{\operatorname{deg}_{T}\left(v_{k}\right)-1} x_{f_{k+1}^{(\alpha)}}^{\epsilon} \\
&=1+b_{v_{k}}(\epsilon)-\sum_{\alpha=1}^{\operatorname{deg}_{T}\left(v_{k}\right)-1}\left(\sum_{w \text { is a woman after } v_{v}^{(\alpha)}}^{\left(\text {(including } v_{k+1}^{(\alpha)}\right)}\right.  \tag{16}\\
&\left.=1+\sum_{w} \sum_{\substack{\left(\text { is a man after } \\
\text { (including } v_{k}\right)}} b_{m}(\epsilon)-\sum_{m \text { is a man after } v_{k+1}^{(\alpha)}} b_{m}(\epsilon)\right) \\
& w \text { is a woman after } v_{k}
\end{align*} b_{w}(\epsilon) .
$$

Thus (14) holds for $v_{k}$. For (15), since $v_{k}$ is a woman, we have $\operatorname{deg}_{T}\left(v_{k}\right) \leq 2$. Hence, no branching appears at $v_{k}$, and we deduce that (15) immediately by the fact that (14) holds for $v_{k+1}$ and feasibility. By induction from the leaf to the root, we can verify that both (14) and (15) hold for any edge on $T$.

Notice that we always have $\sum_{m \in M} b_{m}(\epsilon)<b_{w}(\epsilon)$ for any $m \in M, w \in W$. Hence, the cumulative effects on (14) and (15) make the monotonicity property 2 and 1 true, respectively. In fact, if $x_{f_{i}}=x_{f_{j}}=0$ and $i<j$, then $f_{i}$ is closer to $r$ than $f_{j}$. By (15), there are more women after $v_{i}$ than those after $v_{j}$, which makes $x_{f_{i}}^{\epsilon}$ larger, since $\sum_{m \in M} b_{m}(\epsilon)<b_{w}(\epsilon)$ for any $w \in W . x_{f_{i}}^{\epsilon} \leq \frac{1}{2}$ holds because of Claim 43. We can also obtain the second statement of Claim 47 similarly.

Now we have all the ingredients to conclude the proof of Theorem 42. It suffices to show that, if $(B, D)$ is the current Scarf pair, then Scarf's algorithm both on the original bipartite matching polytope following Algorithm $\mathbb{1}$ and on the perturbed polytope (11) will move to the same new Scarf pair $\left(B^{\prime}, D^{\prime}\right)$. We will prove this assuming $u_{1} \in \mathcal{L}$. Similarly to the discussion in Section 5.4, a similar argument settles the case $u_{1} \in \mathcal{M}$.

By induction hypothesis, in both cases the variable entering $B$ is the unique variable contained in $D \backslash B$, and as usual we denote it by $j_{t}$. In the non-degenerate case, we increase $x_{j_{t}}^{\epsilon}$ from 0 to some strictly positive value (because of non-degeneracy), until there is $j_{\ell} \in B$ such that $x_{j_{\ell}}^{\epsilon}=0$. Since (11) is non-degenerate, the choice of $j_{\ell}$ is unique.

From Lemma 28, when $j_{t}$ enters, there are two possible cases:
(I) $e_{j_{t}}$ joins two different trees $T_{1}, T_{2}$ of $\left(V, E_{B}^{v}\right)$, as to form a larger tree $T$ with two loops, or
(II) $e_{j_{t}}$ connects two nodes of a same tree $T_{1}$ of $\left(V, E_{B}^{v}\right)$,
where in both cases $V_{T_{1}}$ contains the separator $v_{i}$. Hence, $T_{1}$ is of type (a).
Suppose we are in case (I). Then by Proposition [23, and using $j_{t} \in E_{D}, T_{2}$ cannot contain a loop at a man. Thus $T_{2}$ can only be of type (b) or type (d). Let $P$ be the path of form (6), connecting the two loops. By Claim 29, when $e_{j_{t}}$ is one of the man-disliked edges, then all edges which decrease their $x$-value are woman-disliked, except the loop $\left(m_{i}, m_{i}\right)$ which is disliked by man $m_{i}$. This also holds for $x^{\epsilon}$, since the sign of changes is independent from the $b$ vector. Moreover, all the decreasing variables will equally chang ${ }^{11}$. Hence, we will select from all the decreasing variables the one that

[^1]smalest value in $x^{\epsilon}$ leave the basis because this variable will be the first one reaching 0 .
If $T_{2}$ is of type (b), then we know that $x_{\left(w_{i \frac{p}{2}}, w_{\frac{p}{2}}\right)}=0$ by Claim 46, and then by Claim47we have $x_{\left(w_{\frac{p}{2}}, w_{i_{\frac{p}{2}}}\right)} \leq \frac{1}{2}$. Thus $e_{j_{\ell}}$ must be woman-disliked because we know that $x_{\left(m_{i}, m_{i}\right)}^{\epsilon}>\frac{1}{2} \geq x_{\left(w_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}\right)}$, so the smallest entry cannot be $x_{\left(m_{i}, m_{i}\right)}^{\epsilon}$.

If $T_{2}$ is of type (d), then $x_{\left(w_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}\right)}=1$ and by feasibility $x_{\left(w_{\left.i_{\frac{p}{2}}, w_{i_{\frac{p}{2}}}\right)}\right.}=1+b_{w_{i_{\frac{p}{2}}}}(\epsilon)>1$. The path $P$ is in fact $x$-augmenting. By the second statement in Claim 47, the smallest decreasing variable in $P$ is $x_{\left(m_{i}, m_{i}\right)}^{\epsilon}$, since $x_{\left(m_{i}, m_{i}\right)}^{\epsilon}<1<x_{\left(w_{\left.i_{\frac{p}{2}}, w_{i \frac{p}{2}}\right)}\right)}$, and by monotonicity we have $x_{\left(m_{i}, m_{i}\right)}^{\epsilon}<x_{e}^{\epsilon}$ for any woman-disliked edge on $E_{P} \cap E_{T_{1}}$.

Therefore, if (I) happens, the leaving edge is selected following the rule of Algorithm 1 ,
If (II) happens, when $e_{j_{t}}$ is man-disliked, then by Claim 30, the decreasing variables are all corresponding to woman-disliked edges, thus $e_{j_{\ell}}$ must be woman-disliked, which also follows the rule of Algorithm $\mathbb{1}$.

## 7 On Expressing Stable Matchings as Dominating Vertices

### 7.1 Failure in Representing the Intermediate Matchings

It is natural to ask which stable matchings can be output using Scarf's algorithm (with any pivoting rule). Under the hypothesis of $C$ being consistent (see Section (2.3) we give a necessary condition for a stable matching to be represented by a dominating basis in the marriage model. Combined with Example 51, our result shows that the idea of expressing the stable matchings as dominating vertices is limited, in the sense that in the worst case, the number of stable matchings is exponentially greater than the number of those corresponding to dominating vertices of $\mathcal{P}_{M}$. Recall that a stable matching $\mu$ is $v$-optimal (see Section (2.3) for some agent $v$ if $v$ is matched in $\mu$ to her best partner among all the partners she is matched to across all stable matchings.

Definition 48 (Intermediate Stable Matching). We call a stable matching $\mu$ intermediate, if there is no $v \in V$ such that $\mu$ is $v$-optimal.

As usual (see the discussion in Section 1.1), we restrict to complete instances with the same number of men and women.

Theorem 49. Suppose we have a marriage instance $\mathcal{I}=(G(V, E), \succ)$ with complete preference lists and same number of men and women. Pick any consistent ordinal matrix C (c.f. Definition 6). If a stable matching $\mu$ is intermediate, then there is no dominating vertex $x$ such that $x$ is the characteristic vector of $\mu$.

Theorem 49 tells us that under the consistency condition, there is no way to represent an intermediate stable matching by any dominating vertex. Thus, any implementation/pivoting rule of Scarf's algorithm will not lead us to such stable matchings. To prove this, we use the following well-known result, see [20].

Lemma 50. Consider a marriage instance $\mathcal{I}$ defined over a graph $G(M \cup W, E)$. Then, there exists two stable matchings $\mu_{0}$ and $\mu_{z}$ such that $\mu_{0}$ (resp. $\mu_{z}$ ) is m-optimal (resp. w-optimal) for any $m \in M$ (resp. $w \in W$ ).

Proof of Theorem 49. Fix a intermediate stable matching $\mu$ (we assume there exists one, otherwise the statement trivially holds). Fix $(A, b, C)$ such that $(A, b)$ defines the matching polytope of the instance and $C$ is consistent. Suppose by contradiction there exists a dominating vertex $x$ for $(A, b, C)$, such that $x$ is the characteristic vector of $\mu$, i.e. $x_{e}=1$ if and only if $e \in \mu$.

Consider any dominating basis $B$ corresponding to the vertex $x$. Notice that if $e_{j} \in \mu$, then $j \in B$. Therefore, then, lity vector associated to the ordinal basis $B$, we have for any $i \in[n]$,

$$
\begin{equation*}
u_{i}=\min _{j \in B} c_{i j} \leq c_{i,(i, \mu(i))} \tag{17}
\end{equation*}
$$

Notice that by Lemma 21, there exists at least one loop in $E_{B}$. Suppose $\left(v_{\ell}, v_{\ell}\right) \in E_{B}$ for some $\ell$. Then, by consistency of $C$, we have

$$
\begin{equation*}
u_{\ell}=\min _{j \in B} c_{\ell j}=c_{\ell,\left(v_{\ell}, v_{\ell}\right)} . \tag{18}
\end{equation*}
$$

Without loss of generality, we assume $v_{\ell} \in M$ is a woman. Let $\mu_{0}$ be the man-optimal stable matching. Consider the edge $e=\left(m^{*}, v_{\ell}\right)$, where $m^{*}=\mu_{0}\left(v_{\ell}\right)$ is the partner of $v_{\ell}$ in $\mu_{0}$. Note that $m^{*}$ exists since we assume that the number of men and women coincide and lists are complete. By Lemma 50, from all stable matchings, $v_{\ell}$ is the best possible partner of $m^{*}$. Denote the column corresponding to $e$ as $c_{e}$, then, by consistency of $C$ and (18),

$$
\begin{equation*}
c_{v_{\ell}, e}>c_{v_{\ell},\left(v_{\ell}, v_{\ell}\right)}=u_{\ell} . \tag{19}
\end{equation*}
$$

Notice that $v_{\ell}$ is the best possible partner among all partners $m^{*}$ is matched to in a stable matching, while $\mu\left(m^{*}\right)$ is less preferred by $m^{*}$ since $\mu$ is intermediate. Combining this observation with (17), we obtain

$$
\begin{equation*}
c_{m^{*}, e}>c_{m^{*},\left(m^{*}, \mu\left(m^{*}\right)\right)} \geq u_{m^{*}} . \tag{20}
\end{equation*}
$$

By the fact that a node $v \neq v_{\ell}, m^{*}$ is not incident to $e$ and the consistency of $C$, we have

$$
\begin{equation*}
c_{v, e}>u_{v}, \forall v \neq v_{\ell}, m^{*} . \tag{21}
\end{equation*}
$$

Now, by (19), (20), and (21), we find a column $c_{e}$ such that $c_{e}>u$, contradicting that $B$ is a dominating basis. Then the thesis follows.

### 7.2 An Example with Exponentially Many Intermediate Matchings

Example 51. We give an infinite family of instances where the $v$-optimal stable matchings form an exponentially smaller subset of the set of all stable matchings.

For $n=2 k \in 2 \mathbb{N}$, consider the instance with men $m_{0}, \ldots, m_{k-1}$, women $w_{0}, \ldots, w_{k-1}$. For $i \in\{0, \ldots, k-1\}$, the preference list of man $m_{i}$ is given by:

$$
m_{i}: w_{i} \succ_{m_{i}} w_{i+1} \succ_{m_{i}} w_{i+\frac{k}{2}+1} \succ_{m_{i}} w_{i+\frac{k}{2}+2},
$$

where indices are taken modulo $k$. Women's lists also have length 4 and are such that woman $w_{j}$ lists man $m_{i}$ in position $\ell$ if and only if man $m_{i}$ lists woman $w_{j}$ in position $5-\ell$. That is, for $i \in\{0, \ldots, k-1\}$ we have:

$$
w_{i}: m_{i-\frac{k}{2}-2} \succ_{w_{i}} m_{i-\frac{k}{2}-1} \succ_{w_{i}} m_{i-1} \succ_{w_{i}} m_{i}
$$

where again indices are taken modulo $n$. See Table $\square$ for an example.

| 0 | 0 | 1 | 6 | 7 | 0 | 3 | 4 | 9 | 0 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | 2 | 7 | 8 | 1 | 4 | 5 | 0 | 1 |
| 2 | 2 | 3 | 8 | 9 | 2 | 5 | 6 | 1 | 2 |
| 3 | 3 | 4 | 9 | 0 | 3 | 6 | 7 | 2 | 3 |
| 4 | 4 | 5 | 0 | 1 | 4 | 7 | 8 | 3 | 4 |
| 5 | 5 | 6 | 1 | 2 | 5 | 8 | 9 | 4 | 5 |
| 6 | 6 | 7 | 2 | 3 | 6 | 9 | 0 | 5 | 6 |
| 7 | 7 | 8 | 3 | 4 | 7 | 0 | 1 | 6 | 7 |
| 8 | 8 | 9 | 4 | 5 | 8 | 1 | 2 | 7 | 8 |
| 9 | 9 | 0 | 5 | 6 | 9 | 2 | 3 | 8 | 9 |

Table 1: The instance constructed in Example 51 for $n=20$. On the left, preference lists or men are given, while on the right, preference lists of women are given.

Lists are incomplete, but it is well-known that one can complete them by adding missing entries at the end of the preference lists, without changing the set of stable matchings, see, e.g., [20]. So the hypothesis from Theorem 49 hold, and we investigate the instance with incomplete lists for ease of exposition. For $i=0, \ldots, k-1$, man $m_{i}$ lists woman $w_{i}$ as his top choice. Hence, the man-optimal stable matching $\mu_{0}$ assigns man $m_{i}$ to woman $w_{i}$. Similarly, the woman-optimal stable matching $\mu_{z}$ assigns each woman their favorite man.

We next prove the required properties for Example 51. The next claim shows that the manand the woman-optimal stable matchings are the the only stable matchings that are $v$-optimal for some $v$.

Claim 52. Let $\mu$ be a $v$-optimal stable matching for some $v \in M \cup W$. Then $\mu \in\left\{\mu_{0}, \mu_{z}\right\}$.
On the other hand, the instance has exponentially many stable matchings.
Claim 53. Let $S \subset\left\{0, \ldots, \frac{k}{2}-1\right\}$. Then the matching that, for $i \in S$, assigns men $i, i+\frac{k}{2}$ to their third-favorite partner and, for $i \in\left\{0, \ldots, \frac{k}{2}-1\right\} \backslash S$, assigns men $i, i+\frac{k}{2}$ their second-favorite partner is stable.

Let us show that Claim 52 and Claim 53 imply the thesis. By Claim 53, the instance has at least $\sqrt{2}^{k}$ stable matchings. On the other hand, by Claim 52 there are exactly 2 stable matchings that are $v$-optimal for some agent $v$.

The proofs of Claim 52 and Claim 53 require the introduction of the classical concept of rotations exposed at a matching, and is given in Appendix A.3.

### 7.3 When $C$ is not consistent

foe in the next example, even if we drop the condition that $C$ is consistent, the broader class of all ordinal matrices $C$ does not help us represent all stable matchings, even if we allow for choices of $C$ such that some of the dominating vertices of $(A, b, C)$ are not stable matchings. We show this fact through the next example that is well-studied in many stable matching contexts, for instance in (14):

Example 54. Consider the following preference lists for 6 agents:
We have three stable matchings: man-optimal $\mu_{1}=\left\{\left(m_{1}, w_{1}\right),\left(m_{2}, w_{2}\right),\left(m_{3}, w_{3}\right)\right\}$, intermediate $\mu_{2}=\left\{\left(m_{1}, w_{2}\right),\left(m_{2}, w_{3}\right),\left(m_{3}, w_{1}\right)\right\}$, and woman-optimal $\mu_{3}=\left\{\left(m_{1}, w_{3}\right),\left(m_{2}, w_{1}\right),\left(m_{3}, w_{2}\right)\right\}$.

| 1 | 1 | 2 | 3 | 1 | 2 | 3 | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 2 | 2 | 3 | 1 | 2 | 3 | 1 | 2 |
| 3 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |

Table 2: The instance constructed in Example 54 for $n=6$. On the left, preference lists or men are given, while on the right, preference lists of women are given.

We drop the condition of consistency on C (c.f. Section 2.3). We want to find an ordinal matrix $\tilde{C}$, such that all dominating vertices of $(A, b, \tilde{C})$ contain the three vertices $x_{1}, x_{2}, x_{3}$ of the polytope $\mathcal{P}_{F M}$ defined by $(A, b)$ such that $x_{i}$ is the characteristic vector of $\mu_{i}$ for $i=1,2,3$. Notice that, we do not even consider how to implement Scarf's algorithm to obtain them. Given the size of the instance, is not hard to do a complete enumeration of all matrices $\tilde{C}$ to conclude that such $\tilde{C}$ does not exist (recall that only the relative ordering of entries of $\tilde{C}$ matters).

## 8 Conclusions and Future Work

Our paper shows what is, to the best of our knowledge, the first proof of polynomial-time convergence of Scarf's algorithm in relevant settings, as well as the first negative results on the expressive power of dominating vertices, hence of approaches that rely on Scarf's result. On one hand, we give supporting evidence that Scarf's algorithm can be proved to run in polynomial time in relevant cases, especially when we can leverage on a combinatorial interpretation of the input. On the other hand, we show that Scarf's algorithm can have structural limits much stronger than those of the search and enumeration problems it is associated to.

Understanding Scarf's algorithm on the bipartite matching polytope foi ordinal matrices $C$ is an appealing theoretical question, although we are not aware of any model employing more general matrices $C$ on the bipartite matching polytope. Two specific questions here are in order. First, although we do not expect non-consistent $C$ to be meaningful for expressing stable matchings as dominating vertices, this statement requires a proof, and it would be useful to have an extension of Theorem 5 to non-consistent matrices. Second, it would be interesting to understand whether Scarf's algorithm converges in polynomial time on the bipartite matching polytope for any ordinal matrix $C$. This would require an understanding of the problem that goes beyond stable marriages.

Future work also include the investigation of Scarf's algorithm in more general settings, as well as the relation between our pivoting rules and classical algorithms in the area, such as Tan's [39] and Roth-Vande Vate's random paths to stability [34].

## Acknowledgments.

Yuri Faenza and Chengyue He acknowledge support from the National Science Foundation grant CAREER: An algorithmic theory of matching markets.

## References

[1] Ron Aharoni and Tamás Fleiner. On a lemma of Scarf. Journal of Combinatorial Theory, Series B, 87(1):72-80, 2003.
[2] Ron Aharoni and Ron Holzman. Fractional kernels in digraphs. J. Combinatorial Theory, 73(1):1-6, 1998.
[3] Michel Louis Balinski. On the maximum matching, minimum covering. In Proc. Symp. Math. Programming, pages 301-312. Princeton University Press, 1970.
[4] Roger E Behrend. Fractional perfect b-matching polytopes I: General theory. Linear Algebra and its Applications, 439(12):3822-3858, 2013.
[5] Dimitris Bertsimas and John N Tsitsiklis. Introduction to linear optimization, volume 6.
[6] Péter Biró and Tamás Fleiner. Fractional solutions for capacitated NTU-games, with applications to stable matchings. Discrete Optimization, 22:241-254, 2016.
[7] Péter Biró, Tamás Fleiner, and Robert W Irving. Matching couples with Scarf's algorithm. Annals of Mathematics and Artificial Intelligence, 77(3):303-316, 2016.
[8] Alexander Black, Jesús De Loera, Sean Kafer, and Laura Sanità. On the simplex method for 0/1 polytopes. arXiv preprint arXiv:2111.14050, 2021.
[9] Robert G Bland. New finite pivoting rules for the simplex method. Mathematics of operations Research, 2(2):103-107, 1977.
[10] Karthekeyan Chandrasekaran, László A Végh, and Santosh S Vempala. The cutting plane method is polynomial for perfect matchings. Mathematics of Operations Research, 41(1):2348, 2016.
[11] Xi Chen and Xiaotie Deng. Settling the complexity of two-player nash equilibrium. In 2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06), pages 261272. IEEE, 2006.
[12] Gergely Csáji. On the complexity of stable hypergraph matching, stable multicommodity flow and related problems. 2021.
[13] Constantinos Daskalakis, Paul W Goldberg, and Christos H Papadimitriou. The complexity of computing a nash equilibrium. SIAM Journal on Computing, 39(1):195-259, 2009.
[14] Piotr Dworczak. Deferred acceptance with compensation chains. Operations Research, 69(2):456-468, 2021.
[15] Jack Edmonds, Stéphane Gaubert, and Vladimir Gurvich. Scarf oiks. Electronic Notes in Discrete Mathematics, 36:1281-1288, 2010. ISCO 2010 - International Symposium on Combinatorial Optimization.
[16] Pavlos Eirinakis, Dimitrios Magos, and Ioannis Mourtos. From one stable marriage to the next: How long is the way? SIAM Journal on Discrete Mathematics, 28(4):1971-1979, 2014.
[17] Pavlos Eirinakis, Dimitrios Magos, Ioannis Mourtos, and Panayiotis Miliotis. Polyhedral aspects of stable marriage. Mathematics of Operations Research, 39(3):656-671, 2014.
[18] Yuri Faenza and Xuan Zhang. Affinely representable lattices, stable matchings, and choice functions. Mathematical Programming, pages 1-40, 2022.
[19] David Gale and Lloyd S Shapley. College admissions and the stability of marriage. The American Mathematical Monthly, 69(1):9-15, 1962.
[20] Dan Gusfield and Robert W Irving. The stable marriage problem: structure and algorithms. MIT press, 1989.
[21] Penny E Haxell and Gordon T Wilfong. A fractional model of the border gateway protocol (BGP). In Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms, pages 193-199, 2008.
[22] Robert W Irving, Paul Leather, and Dan Gusfield. An efficient algorithm for the "optimal" stable marriage. Journal of the ACM (JACM), 34(3):532-543, 1987.
[23] Takashi Ishizuka and Naoyuki Kamiyama. On the complexity of stable fractional hypergraph matching. In 29th International Symposium on Algorithms and Computation (ISAAC 2018). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.
[24] Shiva Kintali, Laura J Poplawski, Rajmohan Rajaraman, Ravi Sundaram, and Shang-Hua Teng. Reducibility among fractional stability problems. SIAM Journal on Computing, 42(6):2063-2113, 2013.
[25] Tamás Király and Júlia Pap. A note on kernels and Sperner's lemma. Discrete Applied Mathematics, 157(15):3327-3331, 2009.
[26] László Lovász and Michael D Plummer. Matching theory, volume 367. American Mathematical Soc., 2009.
[27] David Manlove. Algorithmics of matching under preferences, volume 2. World Scientific, 2013.
[28] Hai Nguyen, Thành Nguyen, and Alexander Teytelboym. Stability in matching markets with complex constraints. Management Science, 67(12):7438-7454, 2021.
[29] Thành Nguyen and Rakesh Vohra. Near-feasible stable matchings with couples. American Economic Review, 108(11):3154-69, 2018.
[30] Thành Nguyen and Rakesh Vohra. Stable matching with proportionality constraints. Operations Research, 67(6):1503-1519, 2019.
[31] James B Orlin. A polynomial time primal network simplex algorithm for minimum cost flows. Mathematical Programming, 78(2):109-129, 1997.
[32] Christos H Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. Journal of Computer and system Sciences, 48(3):498-532, 1994.
[33] Alvin E Roth, Uriel G Rothblum, and John H Vande Vate. Stable matchings, optimal assignments, and linear programming. Mathematics of operations research, 18(4):803-828, 1993.
[34] Alvin E Roth and John H Vande Vate. Random paths to stability in two-sided matching. Econometrica: Journal of the Econometric Society, pages 1475-1480, 1990.
[35] Uriel G Rothblum. Characterization of stable matchings as extreme points of a polytope. Mathematical Programming, 54(1):57-67, 1992.
[36] Laura Sanità. The diameter of the fractional matching polytope and its hardness implications. In 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS), pages 910-921. IEEE, 2018.
[37] Herbert E Scarf. The core of an N person game. Econometrica: Journal of the Econometric Society, pages 50-69, 1967.
[38] Jimmy JM Tan. Stable matchings and stable partitions. International Journal of Computer Mathematics, 39(1-2):11-20, 1991.
[39] Jimmy JM Tan and Hsueh Yuang-Cheh. A generalization of the stable matching problem. Discrete Applied Mathematics, 59(1):87-102, 1995.
[40] Robert E Tarjan. Dynamic trees as search trees via euler tours, applied to the network simplex algorithm. Mathematical Programming, 78(2):169-177, 1997.
[41] Chung-Piaw Teo and Jay Sethuraman. The geometry of fractional stable matchings and its applications. Mathematics of Operations Research, 23(4):874-891, 1998.
[42] John H Vande Vate. Linear programming brings marital bliss. Operations Research Letters, 8(3):147-153, 1989.
[43] Rakesh Vohra. Scarf's lemma and applications, 2019. Talk at Simons Institute, UC Berkeley.

## A Missing proofs

## A. 1 Proof of Lemma 21 and Lemma 22

Proof of Lemma 21, First assume $B$ is a basis, i.e., $\operatorname{rank}(B)=n$. To show the first claim, if $E_{B}^{v}$ contains a cycle, then it is an even cycle. An even cycle implies linear dependence of the corresponding columns in $B$, a contradiction.

Suppose there is a singleton node $v_{i}$, covered neither by a valid edge nor by its loop. Then the row $v_{i}$ of $B$ are all 0 's, which makes $\operatorname{rank}(B) \leq n-1$, a contradiction.

Recall the following linear algebra fact: write $B=\left(\begin{array}{ll}B_{1} & B_{2} \\ B_{3} & B_{4}\end{array}\right)$, and suppose that $B_{4}=0$ and $B_{3}$ has $q$ rows. Then $B_{3}$ has at least $q$ columns 2 . Consider a tree $T$ with at least two nodes (thus at least one valid edge), suppose first a tree of $E_{B}^{v}$ is incident to no loop. Then we can apply the statement above to $B$ by taking the rows of $B_{3}$ to be the node set of $T$, and its columns the edge set of $T$ and obtaining a contradiction. If conversely a tree of $E_{B}^{v}$ is incident to two or more loops, we can apply the statement above by taking the rows of $B_{3}$ to be all nodes except those of $T$, and by columns all columns of $B$ corresponding to edges not incident to nodes in $T$, obtaining again a contradiction.

On the other hand, suppose the two graph conditions are satisfied. It is well-known that an incidence matrix of a tree can be permuted to have the form

$$
\begin{array}{r}
v_{i_{1}} \\
v_{i_{2}} \\
v_{i_{3}} \\
\vdots \\
v_{i_{t+1}}
\end{array}\left(\begin{array}{cccc}
e_{j_{1}} & e_{j_{2}} & \cdots & e_{j_{t}} \\
1 & * & & * \\
1 & * & & * \\
0 & 1 & & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & & 1
\end{array}\right)
$$

[^2]with arbitrary node $v_{i_{1}}$ to be the first row (this fact can be shown by induction), and each column having exactly two 1 s . We can extend this fact to our tree with a loop $e_{j_{t+1}}$ incident, w.l.o.g., to $v_{i_{1}}$. After adding the loop column, the new $t \times t$ square submatrix has determinant 1 , implying full rank. We can apply this to every tree in our forest and obtain that $B$ has full rank, thus a basis.

Proof of Lemma 20. By Lemma 21, suppose that the edges $E_{B}$ can be decomposed into $\tau$ connected components. Then for every $\omega \in[\tau]$, consider the $\omega$-th component, its local incidence matrix corresponds to a square submatrix $B_{\omega}$, up to permuting rows and columns. Thus we obtain the desired result.

## A. 2 Proof of Lemma 39

Proof. Let $(B, D)$ be the first Scarf pair with $u_{1} \in \mathcal{M}$ obtained by applying Algorithm 1 We have already argued that, in the Scarf pair preceding $(B, D)$, we have $u_{1} \in \mathcal{L}$. We can then apply Lemma 37 part 2(ii) and deduce that (i),(ii) from Lemma 39 are satisfied, and $e_{j_{t}}$ is a man-disliked valid edge.

Let now $(B, D)$ be a generic Scarf pair visited by the algorithm, and assume it verifies $u_{1} \in \mathcal{M}$, (i), (ii) from Lemma 39 and that $e_{j_{t}}$ is a man-disliked valid edge. We show that $(B, D)$ satisfies (iii) from Lemma 39, As a consequence of (iii), the algorithm either continues or terminates. If it continues, we obtain (iv) and that the next pair $\left(B^{\prime}, D^{\prime}\right)$ visited by the algorithm satisfies $u_{1}^{\prime} \in \mathcal{M}$, (i),(ii) from Lemma 39 and $e_{j^{*}}$ (where $\left\{j^{*}\right\}=D^{\prime} \backslash D$ ) is a man-disliked valid edge. Else, if we terminate, then we obtain $B^{\prime}=D$ (i.e., $(v)$ is verified). This concludes the proof.

We start with auxiliary claims. Recall that the basis structure (i.e., Lemma 21) is independent of the ordinal basis and the utility vector. Consider the tree $T$ (with single loop, but we omit to repeat "with single loop" below) of $G_{B}$ containing $m_{1}$. Unlike the case $u_{1} \in \mathcal{L}$, now $T$ is no longer a singleton, since we know from (ii) there is a valid edge $\left(m_{1}, w\right) \in E_{D}$.

Claim 55. The entering edge $e_{j_{t}}\left(\notin E_{B}\right)$ is incident to $T$. In other words, $e_{j_{t}}$ and $T$ have at least one common node.

Proof of Claim 55, Since $e_{j_{t}} \notin E_{B}$, using Lemma 21, $e_{j_{t}}$ will either connect two trees or form an even cycle in one tree. Both cases create a connected component $\Gamma$ in $E_{B \cup\left\{e_{j_{t}}\right\}}$ with the number of edges being one more than the number of nodes. Since $e_{j_{t}} \in E_{\Gamma}$, it suffices to show that $m_{1}$ is on $\Gamma$.

If $m_{1}$ is not on $\Gamma$, then all the edges on $\Gamma$ are contained in $E_{D}$ since $D=B \cup\left\{j_{t}\right\} \backslash\{1\}$. By Corollary [25, any edge in $E_{D}$ is disliked in $D$ by either $m_{1}$ or one of its endpoints. Notice that on $\Gamma$ no edge can be disliked in $D$ by $m_{1}$ since we have the induction hypothesis (ii). Thus, any edge on $\Gamma$ is disliked in $D$ by one of the nodes also on $\Gamma$, which is impossible since the number of edges does not match the number of nodes on $\Gamma$.

The following shows that Lemma 26 holds even though we do not have $u_{1} \in \mathcal{L}$.
Claim 56. Either $B^{\prime}=D$ or there exists $\bar{w} \in W$ such that $(\bar{w}, \bar{w}) \in E_{B^{\prime}} \cap E_{D^{\prime}}$ and $\bar{w}$ is not properly matched in $\mu_{B^{\prime}}$, where $\mu_{B^{\prime}}$ is the matching corresponding to the feasible basis $B^{\prime}$.

Proof of Claim 56] Let $x, x^{\prime}$ be the basic feasible solution associated with the basis $B, B^{\prime}$, respec$\overline{\text { tively. Assume } B^{\prime}} \neq D$. If $x_{\left(m_{1}, m_{1}\right)}^{\prime}=1$, then we know the matching $\mu_{B}^{\prime}$ does not properly match all agents. Thus, there exists some woman $\bar{w}$ who is also not properly matched. Hence, $x_{(\bar{w}, \bar{w})}^{\prime}=1$ and $(\bar{w}, \bar{w}) \in E_{B^{\prime}}$. Since $B^{\prime} \neq D$, we have $B^{\prime} \backslash\{1\} \subseteq D^{\prime}$, which implies $(\bar{w}, \bar{w}) \in E_{B^{\prime}} \cap E_{D^{\prime}}$. It therefore suffices to show that $x_{\left(m_{1}, m_{1}\right)}^{\prime}=1$.

Now suppose that we are at the first iteration such that $u_{1} \in \mathcal{M}$ and $x_{\left(m_{1}, m_{1}\right)}^{\prime}=0$. Then we have $x_{\left(m_{1}, m_{1}\right)}=1$ and $x_{\left(m_{1}, m_{1}\right)}^{\prime}=0$, i.e., a non-degenerate cardinal pivot makes the value $x_{\left(m_{1}, m_{1}\right)}$ decrease. We will show that by Algorithm $x_{\left(m_{1}, m_{1}\right)}$ is a leaving variable, hence, by Lemma 14 , $B^{\prime}=D$, a contradiction.

Let $\Gamma$ be the connected component of $G_{B \cup\left\{j_{t}\right\}}$ that contains $e_{j_{t}}$.
If $\Gamma$ contains an even cycle, then by Claim $55 \Gamma$ contains exactly one loop ( $m_{1}, m_{1}$ ). Since the pivoting is non-degenerate, $x_{j_{t}}$ changes from 0 to 1 . Since $e_{j_{t}}$ belongs to an even cycle, the cycle must be $x$-alternating, and the change of weights only happen inside the cycle. There is no way to change $x_{\left(m_{1}, m_{1}\right)}$ from 1 to 0 without violating the feasibility. Thus, $x_{\left(m_{1}, m_{1}\right)}^{\prime}=0$ is impossible.

If, on the other hand, $e_{j_{t}}$ joins two different trees in $G_{B}$, then there are two loops on $\Gamma$. The change of weight appearing at $e_{j_{t}}$ will lead to the changes of weights on the $x$-augmenting path on $\Gamma$, which results in the change of weights on the loops. Therefore, $x_{\left(m_{1}, m_{1}\right)}^{\prime}=0$, and by Algorithm 1 $B^{\prime}=D$.

Hence, once $x_{\left(m_{1}, m_{1}\right)}=1$ at some iteration, we have $x_{\left(m_{1}, m_{1}\right)}^{\prime}=1$ at all future iterations until termination. Since 1 is therefore not matched, there must be a woman $\bar{w}$ that is not matched, concluding the proof.

To show the correctness of (iii), we can apply the above results and repeat the same arguments in Lemma 28 with two subtle modifications as follows:

First, for formula (6) in the proof:

$$
P=\left(m_{i_{1}}, m_{i_{1}}\right),\left(m_{i_{1}}, w_{i_{1}}\right),\left(w_{i_{1}}, m_{i_{2}}\right), \ldots,\left(m_{i_{\frac{1}{2}}}, w_{i_{\underline{2}}}\right),\left(w_{i_{\underline{2}}}, w_{i_{\underline{2}}}\right) .
$$

We assign $m_{i_{1}}=m_{1}$, and $w_{i_{\underline{p}}}=\bar{w}$. Then the edges of $P$ are disliked by

$$
\text { none, } m_{1}, w_{i_{1}}, m_{i_{2}}, w_{i_{2}}, \ldots, m_{i_{\frac{p}{2}-1}}, w_{i \frac{p}{2}-1}, m_{i \frac{p}{2}}, w_{i \frac{p}{2}}
$$

in order (notice that, $\left(m_{i_{1}}, m_{i_{1}}\right)=\left(m_{1}, m_{1}\right)$ does not exist in $E_{D}$ as $1 \notin D$, thus is denoted by "none"-disliked).

Second, for formula (7) in the cycle case when $Q$ contains $m_{1}$ :

$$
Q=\left(m_{i_{1}}, w_{i_{1}}\right),\left(w_{i_{1}}, m_{i_{2}}\right), \ldots,\left(m_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}\right),\left(w_{i_{\frac{p}{2}}}, m_{i_{1}}\right) .
$$

Here $m_{i_{1}}=m_{1}$ and $w_{i_{1}}=w$ where $w$ is the woman defined in Lemma 39(ii). The edges on $Q$ are still disliked by $m_{1}, w_{i_{1}}, m_{i_{2}}, w_{i_{2}}, \ldots, m_{i_{\frac{p}{2}-1}}, w_{i_{\frac{p}{2}-1}}, m_{i_{\frac{p}{2}}}, w_{i_{\frac{p}{2}}}$ in this order.

Use the remaining arguments in the proof of Lemma 28, we can deduce (iii).
To prove (iv), suppose $e_{j_{\ell}}$ is woman-disliked w.r.t. $D$. Then $v_{i_{\ell}}$ is a woman with $c_{i_{\ell}, j_{\ell}}<c_{i_{\ell}, j_{r}} \in$ $\mathcal{M}$. Thus $e_{j_{r}}$ is a valid edge and the other endpoint corresponds to a man, which is $v_{i_{r}}$. Then the utility of $v_{i_{\ell}}$ increases and that of $v_{i_{r}}$ decreases.

We still need to show that $e_{j^{*}}$ is a man-disliked valid edge w.r.t. $D^{\prime}$. It is man-disliked since $v_{i_{r}}$ is a man. Now suppose $e_{j^{*}}$ is a loop, then $e_{j^{*}}=\left(v_{i_{r}}, v_{i_{r}}\right)$. If $v_{i_{r}}=m_{1}$, we will terminate the algorithm at $\left(B^{\prime}, D^{\prime}\right)$ since $1 \in D^{\prime}$. However, as the loop $\left(m_{1}, m_{1}\right)$ does not leave $E_{B}$, we know $x_{\left(m_{1}, m_{1}\right)}^{\prime}=1$ by our pivoting rule, which implies that $m_{1}$ is not properly matched by $\mu_{B^{\prime}}$, thus $\mu_{B^{\prime}}$ is not a stable matching, a contradiction with Theorem 3. Thus $v_{i_{r}} \neq m_{1}$. Since $v_{i_{r}}$ is a man, $\left(v_{i_{r}}, \bar{w}\right)$ is a valid edge. If $\left(v_{i_{r}}, \bar{w}\right) \notin E_{D^{\prime}}$, then the corresponding column $c_{\left(v_{i_{r}}, \bar{w}\right)}$ is strictly greater than $u^{\prime}$. Else if $\left(v_{i_{r}}, \bar{w}\right) \in E_{D^{\prime}}$, then this edge cannot be disliked by anyone in $D^{\prime}$, since $v_{i_{r}}$ and $\bar{w}$ dislike their loops, and $m_{1}$ dislikes some edge incident to $m_{1}$. Both cases contradict with the definition of the dominating basis.

Since the entering edge $e_{j^{*}}$ can never be a loop, we will maintain (i) for ( $B^{\prime}, D^{\prime}$ ). Also, we know that $u_{1}$ never increases by (iv), thus $v_{i_{\ell}}$ is a woman. Therefore (ii) also holds for ( $B^{\prime}, D^{\prime}$ ).

When 1 leaves $B$, the algorithm terminates. If we are at (v), then we stop the induction.

## A. 3 Missing proofs from Section 7

In order to prove Claim 52 and Claim 53, let us introduce the classical concept of rotations and related properties. For an extensive treatment of rotations, as well as proofs of basic facts on rotations stated here, we refer to [20]. Given a stable matching $\mu$, a $\mu$-alternating cycle is a cycle (in the classical sense) whose edges alternatively belong to $\mu$ and to $E \backslash \mu$. Let

$$
\begin{equation*}
\rho=m_{i_{0}}, e_{0}, w_{i_{0}^{\prime}}, e_{0}^{\prime}, m_{i_{1}}, e_{1}, w_{i_{1}^{\prime}}^{\prime}, \ldots, m_{i_{\ell-1}}, e_{\ell}, w_{i_{\ell-1}^{\prime}} \tag{22}
\end{equation*}
$$

be a $\mu$-alternating cycle. We say that $\rho$ is a rotation exposed at $\mu$ if, for $j=0, \ldots, \ell-1$,

1. $m_{i_{j}}$ is matched to $w_{i_{j}}$ in $\mu$, and
2. $w_{i_{j+1}}$ is the first woman according to $m_{i_{j}}$ 's preference list who prefers $m_{i_{j}}$ to her partner in $\mu$, where indices are taken modulo $\ell$.

Rotations allow us to move from a stable matching to the other. More formally, by defining the symmetric difference operator $\Delta$, the following holds.

Lemma 57. Let $\rho$ be a rotation exposed at some stable matching $\mu$. Then $\mu^{\prime}:=\mu \Delta E_{\rho}$ is a stable matching. Moreover, for each man $m \in M$, either $m$ is matched to the same woman in $\mu$ and $\mu^{\prime}$, or he prefers his assigned partner in $\mu$ to his assigned partner in $\mu^{\prime}$.

Lemma 58. Let $\mu$ be a stable matching. Then there is a sequence of stable matchings $\mu_{0}, \mu_{1}, \ldots, \mu_{s}$ such that:

1. $\mu_{0}$ is the man-optimal stable matching, while $\mu_{s}=\mu$;
2. for $i=1, \ldots$, s, there exists a rotation $\rho_{i}$ exposed at $\mu_{i-1}$ such that $\mu_{i}=\mu_{i-1} \Delta E_{\rho_{i}}$.

Let us now prove an intermediate fact.
Claim 59. The only rotation exposed at $\mu_{0}$ is

$$
\rho_{0}=m_{0}, w_{0}, m_{1}, w_{1}, \ldots, m_{k-1}, w_{k-1} .
$$

Moreover, $\mu^{\prime}:=\mu \Delta E_{\rho_{0}}$ is a stable matching where all men are matched to the second woman in their list.

Proof of Claim [59. Since the instance has the stable matching $\mu_{z} \neq \mu_{0}$, there must be at least one rotation $\rho$ as in (22) exposed in $\mu$ by Lemma 58, Assume w.l.o.g. that $i_{0}=i$ for some $i=0, \ldots, k-1$. Then, by definition of rotation, $i_{0}^{\prime}=i$ and $i_{1}^{\prime}=i+1$. Consequently, $i_{1}=i+1$. Iterating we deduce, $\rho=\rho_{0}$. Moreover, all men and all women are contained in $\rho$. Since all rotations exposed at a stable matching are vertex-disjoint (see again [20]), $\rho_{0}$ is the only rotation exposed at $\mu_{0}$.

Stability of $\mu^{\prime}$ follows from Lemma 57 and the fact that each man is matched to the second woman in his list by the definition of symmetric difference.

We are now ready to prove Claim 52 and Claim 53 .
Proof of Claim 52. By Lemma 58, all stable matchings can be obtained from $\mu_{0}$ by iteratively taking the symmetric difference with rotations from a sequence, and each partial sequence of symmetric differences creates a stable matching. By Claim 59, the only rotation exposed in $\mu_{0}$ is $\rho_{0}$. Hence, all stable matchings other than $\mu_{0}$ can be obtained from $\mu^{\prime}$ by a sequence of rotations eliminations. By Lemma 57, no man improves their partner when the symmetric difference with an exposed rotation
is taken. Hence, in no stable matching other than $\mu_{0}$ a man has a partner he prefers to his partner in $\mu^{\prime}$. Using again Claim 59, we know that $\mu^{\prime}$ is not $v$-optimal for any $v$. We conclude that $\mu_{0}$ is the only $m$-optimal matching for any man $m$.

To conclude the proof of the claim, it suffices to show that $\mu_{z}$ is the only $w$-optimal stable matching for any woman $w$. This follows by observing that

$$
m_{i} \leftrightarrow w_{i+\frac{k}{2}+2} \text { for } i=0, \ldots, k-1
$$

defines a bijection between our class of instances and the instances where role of men and women as swapped (as usual, indices are taken modulo $k$ ). That is, the preference lists of the new instance are, for $i \in\{0, \ldots, k-1\}$ :

$$
w_{i}: m_{i} \succ m_{i+1} \succ m_{i+\frac{k}{2}} \succ m_{i+\frac{k}{2}+1} \quad \text { and } \quad m_{i}: w_{i-\frac{k}{2}-2} \succ w_{i-\frac{k}{2}-1} \succ w_{i-1} \succ w_{i} .
$$

The thesis then follows by the first part of the proof.
Proof of Claim 53. Consider again matching $\mu^{\prime}$ defined in Claim 59. For $i=0, \ldots, \frac{k}{2}-1$, we have that

$$
\rho_{i}:=m_{i}, w_{i+1}, m_{i+\frac{k}{2}}, w_{i+\frac{k}{2}+1}
$$

is a rotation exposed in $\mu^{\prime}$. It is well-known (see again [20]) that if $\rho, \rho^{\prime}$ are rotations exposed in a stable matching $\mu$, then $\rho^{\prime}$ is a rotation exposed in $\mu \Delta \rho$. Hence, for each $S$ as in the hypothesis of the claim, we have that $\left(\left(\left(\mu^{\prime} \Delta \rho_{i_{1}}\right) \Delta \rho_{i_{2}}\right) \ldots \Delta \rho_{i_{|S|}}\right)$ is a stable matching, where $S=\left\{\rho_{i_{1}}, \rho_{i_{2}}, \ldots, \rho_{i_{|S|}}\right\}$. Clearly, all such matchings are distinct. The statement follows.


[^0]:    *IEOR Department, Columbia University, yf2414@columbia.edu
    ${ }^{\dagger}$ IEOR Department, Columbia University, ch3480@columbia.edu
    ${ }^{\ddagger}$ IEOR Department, Columbia University, jay@ieor.columbia.edu

[^1]:    ${ }^{1}$ The cardinal pivot maintains feasibility in the sense of $A_{B} x_{B}+A_{j_{t}} x_{j_{t}}=b+b(\epsilon)$, thus $x_{B}=A_{B}^{-1}(b+b(\epsilon))-$ $A_{B}^{-1} A_{j_{t}} x_{j_{t}}$. Therefore, $x_{j_{i}}$ will change as $x_{j_{t}}$ changes with derivative $\left(A_{B}^{-1} A_{j_{t}}\right)_{j_{i}}$. Since both $A_{B}^{-1}$ and $A_{j_{t}}$ are totally unimodular matrices, if the derivative is negative, it can only be -1 , thus all decreasing variable will change with the same speed.

[^2]:    ${ }^{2}$ This fact follows by subadditivity of the rank, since otherwise $n=r k(B) \leq r k\left(B_{1}, B_{2}\right)+r k\left(B_{3}\right)+r k\left(B_{4}\right)<$ $n-q+q+0=n$, a contradiction.

